Math 22A
Kouba
Solving the Linear Systems $A\vec{x} = \vec{b}$ Using an LU-Decomposition

<u>PROBLEM 1:</u> Solve the following system using an LU-Decomposition:

$$\begin{cases} 2x_1 + 6x_2 = -2 \\ 3x_1 - x_2 = 7 \end{cases} \implies \begin{pmatrix} 2 & 6 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} \implies A\vec{x} = \vec{b}$$

<u>FIRST FIND the LU-DECOMPOSITION for MATRIX A</u>

$$A = \begin{pmatrix} 2 & 6 \\ 3 & -1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 \\ 0 & -10 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1/10 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

so $U = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is Upper Triangular. Then

$$\begin{aligned} L = E_1^{-1}\, E_2^{-1}\, E_3^{-1} &= \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 3 & -10 \end{pmatrix}, \end{aligned}$$

so $L = \begin{pmatrix} 2 & 0 \\ 3 & -10 \end{pmatrix}$ is Lower Triangular, and $A = LU$.

NOW SOLVE the SYSTEM $A\vec{x} = LU\vec{x} = L(U\vec{x}) = \vec{b}$ for $\vec{x}$

STEP I.) First Let $U\vec{x} = \vec{y}$ and Solve $L\vec{y} = \vec{b}$ for $\vec{y}$ $\Longrightarrow$

$$\begin{pmatrix} 2 & 0 \\ 3 & -10 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} \quad \Longrightarrow$$

$$\begin{cases} 2y_1 = -2 \\ 3y_1 - 10y_2 = 7 \end{cases} \quad (\textit{Forward Substitution}) \quad \Longrightarrow$$

$$y_1 = -1 \quad \Longrightarrow \quad 3(-1) - 10y_2 = 7 \quad \Longrightarrow \quad 10y_2 = -10 \quad \Longrightarrow \quad y_2 = -1 \quad \Longrightarrow$$

$$\vec{y} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

STEP II.) Now Solve $U\vec{x} = \vec{y}$ for $\vec{x}$ $\Longrightarrow$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \Longrightarrow$$

$$\begin{cases} x_1 + 3x_2 = -1 \\ x_2 = -1 \end{cases} \quad (\textit{Back Substitution}) \quad \Longrightarrow$$

$$x_1 + 3(-1) = -1 \quad \Longrightarrow \quad x_1 = 2 \quad \Longrightarrow$$

$$\vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

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PROBLEM 2: Solve the following system using an LU-Decomposition:

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 + 8x_3 = -1 \\ -x_1 + 3x_2 + 2x_3 = -2 \end{cases} \quad \Longrightarrow \quad \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 8 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \Longrightarrow \quad A\vec{x} = \vec{b}$$

FIRST FIND the LU-DECOMPOSITION for MATRIX A

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 8 \\ -1 & 3 & 2 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 6 \\ -1 & 3 & 2 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 6 \\ 0 & 2 & 3 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

so $U = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ is Upper Triangular. Then

$$L = E_1^{-1}\ E_2^{-1}\ E_3^{-1}\ E_4^{-1}\ E_5^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 2 & -1 \end{pmatrix},$$

so $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 2 & -1 \end{pmatrix}$ is Lower Triangular, and $A = LU$.

<u>NOW SOLVE the SYSTEM $A\vec{x} = LU\vec{x} = L(U\vec{x}) = \vec{b}$ for $\vec{x}$</u>

STEP I.) First Let $U\vec{x} = \vec{y}$ and Solve $L\vec{y} = \vec{b}$ for $\vec{y}$ $\implies$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \implies$$

$$\begin{cases} y_1 = 1 \\ 2y_1 + 3y_2 = -1 \\ -y_1 + 2y_2 - y_3 = -2 \end{cases} \qquad (\textit{Forward Substitution}) \implies$$

$$y_1 = 1 \implies 2(1) + 3y_2 = -1 \implies 3y_2 = -3 \implies y_2 = -1$$

$$\textit{and} \quad -(1) + 2(-1) - y_3 = -2 \implies y_3 = -1 \implies$$

$$\vec{y} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

STEP II.) Now Solve $U\vec{x} = \vec{y}$ for $\vec{x}$ $\implies$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \implies$$

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_2 + 2x_3 = -1 \\ x_3 = -1 \end{cases} \qquad (\textit{Back Substitution}) \implies$$

$$x_3 = -1 \implies x_2 + 2(-1) = -1 \implies x_2 = 1 \implies$$

$$x_1 - (1) + (-1) = 1 \implies x_1 = 3 \implies$$

$$\vec{x} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

NOTES on LU-DECOMPOSITION :

I.) Alan Turing (1912-1954), a British Mathematician and the "Father of Computer Science", is credited with inventing LU-Decomposition.

II.) Assume that A is a "large" $n \times n$ matrix and $A = LU$ is its LU-Decomposition. It is estimated that the number of "floating point operations" to solve $A\vec{x} = \vec{b}$ using Gauss Elimination (Row Echelon Form) is approximately $(2/3)n^3$, and the the number of "floating point operations" to solve $LU\vec{x} = \vec{b}$ is approximately n^2. This means that it takes approximately

$$\frac{(2/3)n^3}{n^2} = (2/3)n$$

times more operations to solve $A\vec{x} = \vec{b}$ than it does to solve $LU\vec{x} = \vec{b}$.

III.) To save storage space on the computer, the $n \times n$ matrices L and U can be stored as a single matrix B. For example,

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -2 & 1 & 4 \end{pmatrix} \quad \textit{and} \quad U = \begin{pmatrix} 1 & -1 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

can be stored as

$$B = \begin{pmatrix} 2 & -1 & 7 \\ 3 & -1 & 2 \\ -2 & 1 & 4 \end{pmatrix}$$

Conversely, if

$$B = \begin{pmatrix} 3 & 4 & -5 \\ 2 & -1 & -2 \\ 1 & 3 & 2 \end{pmatrix}$$

then it follows that

$$L = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \quad and \quad U = \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

IV.) Later we will learn about the Determinant of a matrix. If an $n \times n$ matrix A can be written as $A = LU$, then computing it's Determinant is extremely easy.