Math 22A
Kouba
Least Squares Solutions to an Unsolvable Linear System $A\vec{x} = \vec{b}$

DEFINITION: The vector space R^n together with the "dot product" operation is called a **real inner product space.**

RECALL: Let A be an $m \times n$ matrix.

I.)

$$dimension(column\ space\ of\ A) = dimension(row\ space\ of\ A)$$

$$= dimension(column\ space\ of\ A^T)$$

$$= dimension(row\ space\ of\ A^T)$$

II.)

$$column\ space\ of\ A = row\ space\ of\ A^T$$

FACTS:

$$(null\ space\ of\ A) \perp (row\ space\ of\ A) \quad \longrightarrow$$

$$(null\ space\ of\ A^T) \perp (row\ space\ of\ A^T) \quad \longrightarrow$$

$$(\#) \qquad (null\ space\ of\ A^T) \perp (column\ space\ of\ A) \quad (by\ II.)$$

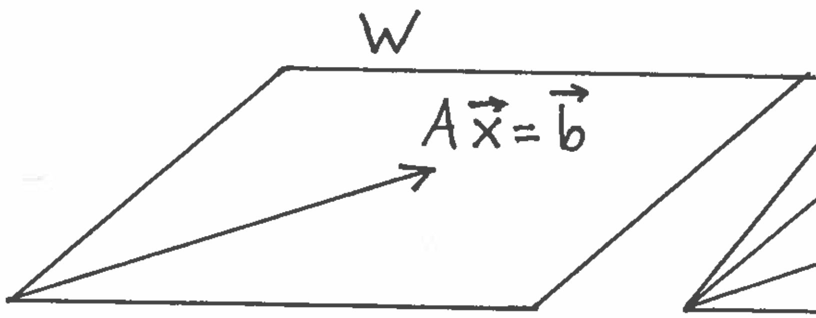
From Fact (#) we can conclude the following:

$$If\ \vec{z} \in (column\ space\ of\ A)^{\perp} \quad \longrightarrow$$

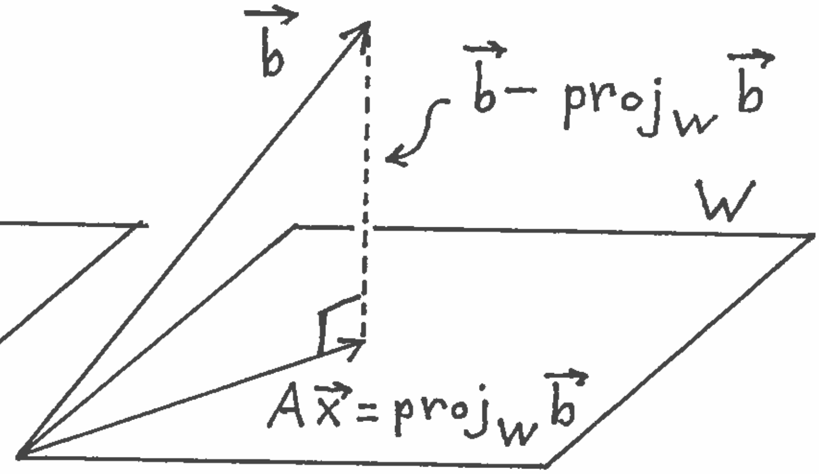
$$\vec{z} \in (null\ space\ of\ A^T) \quad \longrightarrow$$

$$A^T\vec{z} = \vec{0}$$

RECALL: A linear system of equations $A\vec{x} = \vec{b}$ is solvable iff $\vec{b} \in (column\ space\ of\ A)$. Thus, if the system $A\vec{x} = \vec{b}$ is NOT solvable, then $\vec{b} \notin (column\ space\ of\ A)$. Let $W = (column\ space\ of\ A)$.



$A\vec{x} = \vec{b}$ is solvable



$A\vec{x} = \vec{b}$ is NOT solvable

We can see from the diagram that for the case where $A\vec{x} = \vec{b}$ is NOT solvable, the "best wrong answer" is $A\vec{x} = proj_w\vec{b}$, the orthogonal projection of $\vec{b}$ onto W, since $proj_w\vec{b}$ is the vector in $W = (column\ space\ of\ A)$ which is "nearest" $\vec{b}$.

<u>RECALL:</u> The solution $\vec{x}$ to the equation $A\vec{x} = proj_w\vec{b}$ is called the **Least Squares Solution** to the unsolvable system $A\vec{x} = \vec{b}$. The vector $\vec{b} - proj_w\vec{b} = \vec{b} - A\vec{x}$ is called the **Least Squares Error Vector** and $\|\vec{b} - proj_w\vec{b}\| = \|\vec{b} - A\vec{x}\|$ is called the **Least Squares Error**.

We need a process to easily find the Least Squares Solution $\vec{x}$: We know that $A\vec{x} = proj_w\vec{b}$ and that $A\vec{x} \in (column\ space\ of\ A)$, so that

$$\vec{b} - A\vec{x} \in (column\ space\ of\ A)^{\perp} \quad \longrightarrow$$

$$\vec{b} - A\vec{x} \in (null\ space\ of\ A^T) \quad \longrightarrow$$

$$A^T(\vec{b} - A\vec{x}) = \vec{0} \quad \longrightarrow$$

$$A^T\vec{b} - A^TA\vec{x} = A^T\vec{0} = \vec{0} \quad \longrightarrow$$

$$(LS) \qquad A^TA\vec{x} = A^T\vec{b} \quad .$$

Memorize equation (LS) and use itsolve it for $\vec{x}$, the Least Squares Solution to the unsolvable system $A\vec{x} = \vec{b}$. We have proved the following theorem.

THEOREM 6.4.2 *For every linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system*

$$A^TA\mathbf{x} = A^T\mathbf{b} \tag{5}$$

is consistent, and all solutions of (5) are least squares solutions of $A\mathbf{x} = \mathbf{b}$. Moreover, if W is the column space of A, and $\mathbf{x}$ is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of $\mathbf{b}$ on W is

$$\text{proj}_W \mathbf{b} = A\mathbf{x} \tag{6}$$

THEOREM 6.4.3 *If A is an $m \times n$ matrix, then the following are equivalent.*

(*a*) *The column vectors of A are linearly independent.*

(*b*) *A^TA is invertible.*

THEOREM 6.4.4 *If A is an $m \times n$ matrix with linearly independent column vectors, then for every $m \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution. This solution is given by*

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} \tag{9}$$

Moreover, if W is the column space of A, then the orthogonal projection of $\mathbf{b}$ on W is

$$\text{proj}_W \mathbf{b} = A\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{b} \tag{10}$$