Math 22A
Kouba
Real Vector Spaces

<u>DEFINITION:</u> Let V be any nonempty set of objects on which two operations are defined: vector addition and scalar multiplication. If all of the following statements are satisfied for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all scalars $k, l \in R$, then we call V a **vector space** and its objects **vectors**.

1.) If $\vec{u}, \vec{v} \in V$, then $\vec{u} \oplus \vec{v} \in V$. (Closure under Vector Addition)

2.) $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$. (Commutative Property)

3.) $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$. (Associative Property)

4.) There is a vector $\vec{0} \in V$ so that $\vec{u} \oplus \vec{0} = \vec{0} \oplus \vec{u} = \vec{u}$. It is called the **zero vector**. (Zero Vector Property)

5.) For each $\vec{u} \in V$ there is a vector $\vec{v} \in V$ so that $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u} = \vec{0}$. We write $\vec{v} = -\vec{u}$ and call it the **negative** of $\vec{u}$. (Additive Inverse Property)

6.) If $k \in R$ and $\vec{u} \in V$, then $k\vec{u} \in V$. (Closure under Scalar Multiplication)

7.) $k(\vec{u} \oplus \vec{v}) = k\vec{u} \oplus k\vec{v}$. (Distributive Property)

8.) $(k+l)\vec{u} = k\vec{u} \oplus l\vec{u}$. (Distributive Property)

9.) $k(l\vec{u}) = (kl)\vec{u}$. (Associative Property)

10.) $1\vec{u} = \vec{u}$. (Scalar Identity Property)

<u>EXAMPLE 1:</u> Consider all n-dimensional vectors in R^n under standard vector addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so R^n is a vector space.

<u>EXAMPLE 2:</u> Let $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in R \right\}$ under standard matrix addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

<u>EXAMPLE 3:</u> Let $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in R \text{ and } a, b, c, d > 0 \right\}$ under standard matrix addition and scalar multiplication. Properties 1.) through 3.) and 7.) through 10.) are satisfied, but properties 4.), 5.), and 6.) fail, so V is NOT a vector space.

<u>EXAMPLE 4:</u> Let $V = \left\{ \text{functions } f(t) \,\middle|\, f \text{ is continuous for all values of } t \right\}$ under standard point-wise addition and scalar multiplication. Properties 1.) through 10.) are

satisfied, so V is a vector space.

EXAMPLE 5:
Let $V = \left\{ functions\ f(t) \;\middle|\; f\ is\ continuous\ for\ all\ values\ of\ t,\ and\ f(0) = 1 \right\}$ under standard point-wise addition and scalar multiplication. Properties 1.), 4.), 5.), and 6.) are NOT satisfied, so V is NOT a vector space.

EXAMPLE 6: Let $V = \left\{ \overrightarrow{(x,y)} \;\middle|\; x + y = 0 \right\}$ under standard vector addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 7: Let $V = \left\{ \overrightarrow{(x,y)} \;\middle|\; x + y = 2 \right\}$ under standard vector addition and scalar multiplication. Properties 1.), 4.), 5.), and 6.) are NOT satisfied, so V is NOT a vector space.

EXAMPLE 8: Let $V = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; a, b, c, d, e, f \in R \right\}$ under standard matrix addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 9: Let $V = \left\{ ax^3 + bx^2 + cx + d \;\middle|\; a, b, c, d \in R \right\}$ under standard polynomial addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 10: Let $V = \left\{ ax^2 + bx + c \;\middle|\; a, b, c \in I\ (Integers) \right\}$ under standard polynomial addition and scalar multiplication. Properties 1.) through 5.) are satisfied, but Property 6.) is not satisfied, so V is NOT a vector space.

EXAMPLE 11: Let (Real-Valued Sequences)
$V = \left\{ (x_1, x_2, x_3, \cdots) \;\middle|\; x_i \in R\ for\ all\ i = 1, 2, 3, ... \right\}$ under standard vector addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.

EXAMPLE 12:
Let $V = \left\{ functions\ f(t) \;\middle|\; f\ is\ continuous\ for\ all\ values\ of\ t,\ and\ f(0) = 0\ and\ f(-1) = f(3) \right\}$ under standard point-wise addition and scalar multiplication. Properties 1.) through 10.) are satisfied, so V is a vector space.