

DEFINITION : A matrix of the form $R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is called a *rotation matrix*.

FACT : If $\theta > 0$, then R rotates vectors θ radians counter-clockwise. If $\theta < 0$, then R rotates vectors $|\theta|$ radians clockwise. Note that if $\theta = 0$, then $R = I_2$, the two-by-two identity matrix.

PROOF : (For $\theta > 0$) Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a random vector in two-dimensional space with polar form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$, where r is the length and α is the direction of the vector. We want to investigate the vector $R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$\begin{aligned} R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= R \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} \\ &= \begin{pmatrix} r \cos \alpha \cos \theta - r \sin \alpha \sin \theta \\ r \cos \alpha \sin \theta + r \sin \alpha \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} r(\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\ r(\cos \alpha \sin \theta + \sin \alpha \cos \theta) \end{pmatrix} \\ &= \begin{pmatrix} r \cdot \cos(\alpha + \theta) \\ r \cdot \sin(\alpha + \theta) \end{pmatrix}. \end{aligned}$$

This new vector has length r and direction $\alpha + \theta$. Thus, matrix R has rotated vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ θ radians counter-clockwise.