Math 22A
Kouba
The Wronskian

The Wronskian is a tool for determining if functions in a vector space are linearly independent.

DEFINITION: Let $f_1, f_2, f_3, \cdots, f_n$ be $n-1$ times differentiable functions. The **Wronskian** (named after Jozef Hoene de Wronski) of $f_1, f_2, f_3, \cdots, f_n$ is the determinant

$$W(x) = det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

EXAMPLE 1: The Wronskian of $f_1(x) = x^2 + x$, $f_2(x) = 2x + 3$, and $f_3(x) = 4$ is

$$W(x) = det \begin{pmatrix} x^2+x & 2x+3 & 4 \\ 2x+1 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$= +(2)det \begin{pmatrix} 2x+3 & 4 \\ 2 & 0 \end{pmatrix} - (0)det \begin{pmatrix} x^2+x & 4 \\ 2x+1 & 0 \end{pmatrix} + (0)det \begin{pmatrix} x^2+x & 2x+3 \\ 2x+1 & 2 \end{pmatrix}$$

$$= (2)((2x+3)(0) - (4)(2)) = -16$$

EXAMPLE 2: The Wronskian of $f_1(x) = x^2$, $f_2(x) = x^2 + 1$, and $f_3(x) = 1$ is

$$W(x) = det \begin{pmatrix} x^2 & x^2+1 & 1 \\ 2x & 2x & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

$$= +(1)det \begin{pmatrix} 2x & 2x \\ 2 & 2 \end{pmatrix} - (0)det \begin{pmatrix} x^2 & x^2+1 \\ 2 & 2 \end{pmatrix} + (0)det \begin{pmatrix} x^2 & x^2+1 \\ 2x & 2x \end{pmatrix}$$

$$= (2x)(2) - (2x)(2) = 0$$

EXAMPLE 3: The Wronskian of $f_1(x) = \sin^2 x$ and $f_2(x) = \cos^2 x$ is

$$W(x) = det \begin{pmatrix} \sin^2 x & \cos^2 x \\ 2\sin x \cos x & -2\sin x \cos x \end{pmatrix}$$

$$= -2\cos x \sin^3 x - 2\sin x \cos^3 x$$

$$= -2\sin x \cos x(\sin^2 x + \cos^2 x)$$

$$= -2\sin x \cos x(1) = -2\sin x \cos x$$

THEOREM: Let $f_1, f_2, f_3, \cdots, f_n$ be $n-1$ times differentiable functions. If the Wronskian of these functions is NOT identically zero, i.e., if $W(x) \neq 0$, then $f_1, f_2, f_3, \cdots, f_n$ are linearly independent.

PROOF : Consider the equation

$$(\#) \qquad k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$$

We want to show that these functions are linearly independent, i.e., we want to show that the only solution to equation (#) is $k_1 = k_2 = \cdots = k_n = 0$. Now differentiate equation (#) $n-1$ times, generating the following $n \times n$ homogeneous system of equations with variables $k_1, k_2, \cdots, k_n$:

$$\begin{cases} k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0 \\ k_1 f_1'(x) + k_2 f_2'(x) + \cdots + k_n f_n'(x) = 0 \\ k_1 f_1''(x) + k_2 f_2''(x) + \cdots + k_n f_n''(x) = 0 \\ \qquad \vdots \\ k_1 f_1^{(n-1)}(x) + k_2 f_2^{(n-1)}(x) + \cdots + k_n f_n^{(n-1)}(x) = 0 \end{cases}$$

The coefficent matrix for this system of equations is

$$A = \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

Since the determinant of this matrix A is the Wronskian of the functions, and we are assuming that $det(A) = W(x) \neq 0$, it follows that matrix A is invertible and the homogeneous system has only the trivial solution, i.e., $k_1 = k_2 = \cdots = k_n = 0$. QED

IMPORTANT NOTE: If the Wronskian $W(x) = 0$, then no conclusion can be drawn from the Wronskian Method. The functions could be linearly independent or they could be linearly dependent. See the following two examples.

EXAMPLE 4: Let $f_1(x) = 2x^2 + 4$, $f_2(x) = x^2$, and $f_3(x) = 1$. Show that f_1, f_2, and f_3 are linearly dependent by using constants k_1, k_2, k_3. Show that the Wronskian $W(x) = 0$.

EXAMPLE 5: Let $f_1(x) = x^2$ and $f_2(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$. Show that f_1 and f_2 are linearly independent by verifying that f_1 and f_2 are not multiples of each other. Show that the Wronskian $W(x) = 0$.