The Wronskian is a tool for determining if functions in a vector space are linearly independent.

**Definition:** Let \( f_1, f_2, f_3, \ldots, f_n \) be \( n-1 \) times differentiable functions. The **Wronskian** (named after Jozef Hoene de Wronski) of \( f_1, f_2, f_3, \ldots, f_n \) is the determinant

\[
W(x) = \det \begin{pmatrix}
    f_1(x) & f_2(x) & \cdots & f_n(x) \\
    f'_1(x) & f'_2(x) & \cdots & f'_n(x) \\
    f''_1(x) & f''_2(x) & \cdots & f''_n(x) \\
    \vdots & \vdots & \ddots & \vdots \\
    f^{(n-1)}_1(x) & f^{(n-1)}_2(x) & \cdots & f^{(n-1)}_n(x)
\end{pmatrix}
\]

**Example 1:** The Wronskian of \( f_1(x) = x^2 + x, \ f_2(x) = 2x + 3, \) and \( f_3(x) = 4 \) is

\[
W(x) = \det \begin{pmatrix}
    x^2 + x & 2x + 3 & 4 \\
    2x + 1 & 2 & 0 \\
    2 & 0 & 0
\end{pmatrix}
\]

\[
= +2 \det \begin{pmatrix}
    2x + 3 & 4 \\
    2 & 0
\end{pmatrix} - 0 \det \begin{pmatrix}
    x^2 + x & 4 \\
    2x + 1 & 0
\end{pmatrix} + 0 \det \begin{pmatrix}
    x^2 + x & 2x + 3 \\
    2x + 1 & 2
\end{pmatrix}
\]

\[
= 2(2x + 3)(0) - (4)(2) = -16
\]

**Example 2:** The Wronskian of \( f_1(x) = x^2, \ f_2(x) = x^2 + 1, \) and \( f_3(x) = 1 \) is

\[
W(x) = \det \begin{pmatrix}
    x^2 & x^2 + 1 & 1 \\
    2x & 2x & 0 \\
    2 & 2 & 0
\end{pmatrix}
\]

\[
= +1 \det \begin{pmatrix}
    2x & 2x \\
    2 & 2
\end{pmatrix} - 0 \det \begin{pmatrix}
    x^2 & x^2 + 1 \\
    2x & 2
\end{pmatrix} + 0 \det \begin{pmatrix}
    x^2 & x^2 + 1 \\
    2x & 2x
\end{pmatrix}
\]

\[
= 2(2x)(2) - (2x)(2) = 0
\]

**Example 3:** The Wronskian of \( f_1(x) = \sin^2 x \) and \( f_2(x) = \cos^2 x \) is

\[
W(x) = \det \begin{pmatrix}
    \sin^2 x & \cos^2 x \\
    2 \sin x \cos x & -2 \sin x \cos x
\end{pmatrix}
\]
\[
= -2 \cos x \sin^3 x - 2 \sin x \cos^3 x \\
= -2 \sin x \cos x (\sin^2 x + \cos^2 x) \\
= -2 \sin x \cos x (1) = -2 \sin x \cos x
\]

**THEOREM:** Let \( f_1, f_2, f_3, \ldots, f_n \) be \( n - 1 \) times differentiable functions. If the Wronskian of these functions is NOT identically zero, i.e., if \( W(x) \neq 0 \), then \( f_1, f_2, f_3, \ldots, f_n \) are linearly independent.

**PROOF:** Consider the equation

\[
(\#) \quad k_1f_1(x) + k_2f_2(x) + \cdots + k_nf_n(x) = 0
\]

We want to show that these functions are linearly independent, i.e., we want to show that the only solution to equation \((\#)\) is \( k_1 = k_2 = \cdots = k_n = 0 \). Now differentiate equation \((\#)\) \( n - 1 \) times, generating the following \( n \times n \) homogeneous system of equations with variables \( k_1, k_2, \ldots, k_n \):

\[
\begin{align*}
  k_1f_1(x) + k_2f_2(x) + \cdots + k_nf_n(x) & = 0 \\
  k_1f_1'(x) + k_2f_2'(x) + \cdots + k_nf_n'(x) & = 0 \\
  k_1f_1''(x) + k_2f_2''(x) + \cdots + k_nf_n''(x) & = 0 \\
  \quad \vdots & \quad \vdots \\
  k_1f_1^{(n-1)}(x) + k_2f_2^{(n-1)}(x) + \cdots + k_nf_n^{(n-1)}(x) & = 0
\end{align*}
\]

The coefficient matrix for this system of equations is

\[
A = 
\begin{pmatrix}
  f_1(x) & f_2(x) & \cdots & f_n(x) \\
  f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\
  f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x)
\end{pmatrix}
\]

Since the determinant of this matrix \( A \) is the Wronskian of the functions, and we are assuming that \( \det(A) = W(x) \neq 0 \), it follows that matrix \( A \) is invertible and the homogeneous system has only the trivial solution, i.e., \( k_1 = k_2 = \cdots = k_n = 0 \). QED

**IMPORTANT NOTE:** If the Wronskian \( W(x) = 0 \), then no conclusion can be drawn from the Wronskian Method. The functions could be linearly independent or they could be linearly dependent. See the following two examples.

**EXAMPLE 4:** Let \( f_1(x) = 2x^2 + 4 \), \( f_2(x) = x^2 \), and \( f_3(x) = 1 \). Show that \( f_1, f_2, \) and \( f_3 \) are linearly dependent by using constants \( k_1, k_2, k_3 \). Show that the Wronskian \( W(x) = 0 \).
EXAMPLE 5: Let $f_1(x) = x^2$ and $f_2(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$. Show that $f_1$ and $f_2$ are linearly independent by verifying that $f_1$ and $f_2$ are not multiples of each other. Show that the Wronskian $W(x) = 0$. 