Math 22A (Spring 2018)
Kouba
Exam 2

Your Name : KEY

Your EXAM ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 7 pages, including the cover page.

6. You have until 3:02 p.m. sharp to finish the exam. Failure to stop working when time is called may lead to a deduction in your exam score. Thank you for your cooperation.

7. Page 6 is a blank page. It's additional workspace for Problem 7, if you need it.

8. Scratch paper is not allowed. Please show all of your work on the pages provided.

1.) (12 pts.) Find the cosine of the the angle θ between the vector $\vec{v} = \overrightarrow{(-1,-2,1)}$ and line L given by

$$L: \begin{cases} x = 2 - 3t \\ y = t + 3 \\ z = 2t - 1 \end{cases}$$

The direction vector for L is $\vec{w} = \overrightarrow{(-3,1,2)}$ so

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{3-2+2}{\sqrt{1+4+1}\sqrt{9+1+4}}$$

$$= \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{\sqrt{84}}$$

2.) (12 pts.) Use any method to find the determinant of the following matrix.

$$A = \begin{pmatrix} 6 & 4 & 0 \\ -2 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$

$$\det(A) = +(6)\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - (4)\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} + (0)\begin{vmatrix} -2 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= 6(0-(-1)) - 4(-4-3) + 0$$

$$= 6 + 28 = 34$$

3.) (12 pts.) TRUE or FALSE: Let A and B be $n \times n$ matrices. If A is NOT invertible, then AB is NOT invertible. (Prove that it's TRUE or give a counterexample to show that it's FALSE.)

TRUE: assume A is not invertible $\Rightarrow \det(A) = 0$. Show AB is not invertible. Then

$$\det(AB) = \det(A)\cdot\det(B)$$
$$= (0)\det(B) = 0 \Rightarrow$$

AB is not invertible.

4.) (12 pts.) (Triangle Inequality) Let $\vec{v}$ and $\vec{w}$ be vectors in R^n. Prove that $\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$.

proof: $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$

$= \vec{v}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w}$

$= \|\vec{v}\|^2 + \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{w} + \|\vec{w}\|^2$

$= \|\vec{v}\|^2 + 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$

$\le \|\vec{v}\|^2 + 2|\vec{v}\cdot\vec{w}| + \|\vec{w}\|^2$

$\le \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2$

(by Cauchy-Schwarz)

$= (\|\vec{v}\| + \|\vec{w}\|)^2 \Rightarrow$

$\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$

Q.E.D.

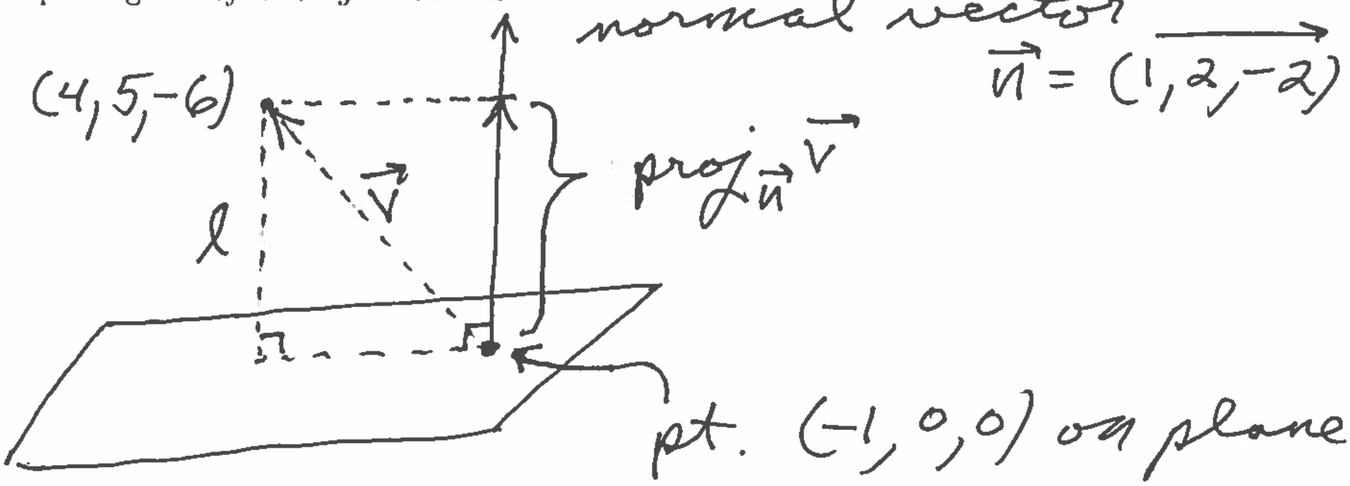
5.)(4 pts. each) Assume that A is a 4×4 matrix and $det(A) = 2$. Find each of the following.

a.) $det(-2A) = (-2)^4 det(A) = 16(2) = 32$

b.) $det(AA^T) = det(A)\cdot det(A^T)$
$= det(A)\cdot det(A) = 2\cdot 2 = 4$

c.) $det((3A)^{-1}) = det(((3I)A)^{-1})$
$= det(A^{-1}(3I)^{-1}) = det(A^{-1}\cdot(\tfrac{1}{3}I))$
$= det(A^{-1})\cdot det(\tfrac{1}{3}I) = \frac{1}{2}\cdot\left(\frac{1}{3}\right)^4 = \frac{1}{162}$

6.) (14 pts.) Use a projection vector to find the distance from the point $(4, 5, -6)$ to the plane given by $x + 2y - 2z = -1$.



$$\text{vector } \vec{v} = \overrightarrow{(4-(-1), 5-0, -6-0)} = \overrightarrow{(5, 5, -6)}$$

$$\Rightarrow l = \| \text{proj}_{\vec{n}} \vec{v} \|$$

$$= \frac{| \vec{v} \cdot \vec{n} |}{\| \vec{n} \|}$$

$$= \frac{5 + 10 + 12}{\sqrt{1+4+4}}$$

$$= \frac{27}{3}$$

$$= 9$$

7.) (14 pts.) Use an LU-decomposition to solve the following system of equations. For full credit find an upper triangular matrix U with 1's along its main diagonal. (THERE IS ADDITIONAL WORK SPACE ON THE NEXT PAGE)

$$\begin{cases} x_1 + x_2 + 2x_3 = 3 \\ 2x_1 + 3x_2 + 2x_3 = 3 \\ 3x_1 + 4x_2 + 5x_3 = 7 \end{cases} \longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{bmatrix} \longrightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 3 & 4 & 5 \end{bmatrix} \longrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(THIS IS ADDITIONAL WORK SPACE FOR PROBLEM 7.)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}, \text{ so}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix};$$ <u>now solve $LU\vec{x} = \vec{b}$</u> :

I.) <u>Solve $L\vec{y} = \vec{b}$ for $\vec{y}$</u> :

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix} \Rightarrow \begin{cases} y_1 = 3 \\ 2y_1 + y_2 = 3 \\ 3y_1 + y_2 + y_3 = 7 \end{cases} \Rightarrow$$

$$y_1 = 3 \Rightarrow 2(3) + y_2 = 3 \Rightarrow y_2 = -3 \Rightarrow$$
$$3(3) + (-3) + y_3 = 7 \Rightarrow y_3 = 1 \Rightarrow \vec{y} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}.$$

II.) <u>Solve $U\vec{x} = \vec{y}$ for $\vec{x}$</u> :

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 = 3 \\ x_2 - 2x_3 = -3 \\ x_3 = 1 \end{cases} \Rightarrow$$

$$x_3 = 1 \Rightarrow x_2 - 2(1) = -3 \Rightarrow x_2 = -1 \Rightarrow$$

$$x_1 + (-1) + 2(1) = 3 \Rightarrow x_1 = 2 \Rightarrow$$

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

8.) (12 pts.) Let $W = \left\{ \overrightarrow{(a, b, c)} \mid a - b = c \right\}$. Show that W is a subspace of R^3.

$W \neq \{\}$ since $\overrightarrow{(4,3,1)} \in W$. assume $\vec{v} = \overrightarrow{(a_1, b_1, c_1)}$, $\vec{w} = \overrightarrow{(a_2, b_2, c_2)} \in W$ and $k \in \mathbb{R}$. Then $a_1 - b_1 = c_1$ and $a_2 - b_2 = c_2$. and

a.) $\vec{v} + \vec{w} = \overrightarrow{(a_1 + a_2, b_1 + b_2, c_1 + c_2)}$ and

$$(a_1 + a_2) - (b_1 + b_2) = a_1 + a_2 - b_1 - b_2$$
$$= (a_1 - b_1) + (a_2 - b_2) = c_1 + c_2 \Rightarrow \vec{v} + \vec{w} \in W.$$

b.) $k\vec{v} = k\,\overrightarrow{(a_1, b_1, c_1)} = \overrightarrow{(ka_1, kb_1, kc_1)}$ and

$$ka_1 - kb_1 = k(a_1 - b_1) = kc_1 \Rightarrow k\vec{v} \in W$$

$\Rightarrow$ W is a subspace of $\mathbb{R}^3$

The following EXTRA CREDIT problem is OPTIONAL. It is worth 10 points.

1.) Let A be an $n \times n$ matrix and assume that $A(A^2 + I) = 2A$. What are all possible values for $det(A)$?

$$A(A^2 + I) = 2A \Rightarrow A^3 + A = 2A \Rightarrow$$
$$A^3 = A \Rightarrow \det(A^3) = \det(A) \Rightarrow$$
$$(\det(A))^3 = \det(A) \Rightarrow$$
$$(\det(A))^3 - \det(A) = 0 \Rightarrow$$
$$\det(A)(\det(A) - 1)(\det(A) + 1) = 0 \Rightarrow$$
$$\det(A) = 0, +1, \text{ or } -1$$