Math 22A (Spring 2018)
Kouba
Exam 3

Your Name : KEY

Your EXAM ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You have until 3:03 p.m. sharp to finish the exam. Failure to stop working when time is called may lead to a deduction in your exam score. Thank you for your cooperation.

7. Scratch paper is not allowed. Please show all of your work on the pages provided.

1.) (8 pts. each) Use any method to determine if the following sets of vectors are linearly independent or linearly dependent.

a.) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \right\}$

$$\begin{vmatrix} 1 & -1 & 4 \\ 2 & 0 & 4 \\ 3 & 1 & 4 \end{vmatrix} = (1)(0-4) - (-1)(8-12) + (4)(2-0)$$

$$= -4 + -4 + 8 = 0,$$ so linearly dependent

b.) $S = \{1, x, \sin x\}$

$$W(x) = \begin{vmatrix} 1 & x & \sin x \\ 0 & 1 & \cos x \\ 0 & 0 & -\sin x \end{vmatrix}$$

$$= (1)(1)(-\sin x) = -\sin x \neq 0,$$ so linearly independent

2.) (12 pts.) Use the Gram-Schmidt Process to convert the following linearly independent set into an orthogonal set.

$$S = \{\overrightarrow{(1,0,0,-1)}, \overrightarrow{(1,1,1,0)}, \overrightarrow{(0,1,-1,1)}\} \quad \longrightarrow$$

$$\vec{v}_1 = \vec{u}_1 = \overrightarrow{(1,0,0,-1)} \quad \longrightarrow$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \overrightarrow{(1,1,1,0)} - \frac{1}{2} \overrightarrow{(1,0,0,-1)}$$

$$= \overrightarrow{\left(\tfrac{1}{2}, 1, 1, \tfrac{1}{2}\right)}, \text{ so let } \vec{v}_2 = \overrightarrow{(1,2,2,1)} \rightarrow$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \overrightarrow{(0,1,-1,1)} - \frac{-1}{2} \overrightarrow{(1,0,0,-1)} - \frac{1}{10} \overrightarrow{(1,2,2,1)}$$

$$= \overrightarrow{\left(\tfrac{1}{2}, 1, -1, \tfrac{1}{2}\right)} - \overrightarrow{\left(\tfrac{1}{10}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{10}\right)}$$

$$= \overrightarrow{\left(\tfrac{2}{5}, \tfrac{4}{5}, \tfrac{-6}{5}, \tfrac{2}{5}\right)}$$

3.) (18 pts.) Let $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 2 & 0 \end{pmatrix}$. Find a basis for the row space of A, a basis for the column space of A, and a basis for the null space of A.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

a.) Basis for row space is

$B = \{ \overrightarrow{(1,0,-3,-1)}, \overrightarrow{(0,1,1,1)} \}$

b.) Basis for column space is

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

c.)
$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 - 3x_3 - x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \Rightarrow$$

let $x_4 = t$ any #, $x_3 = r$ any # $\Rightarrow$
$x_2 = -t - r$, $x_1 = 3r + t$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3r+t \\ -r-t \\ r \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$ so

basis for null space is

$$B = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

5.) (12 pts.) Find the Least Squares Solution and the Least Squares Error for the following unsolvable linear system of equations:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \\ x_1 + 2x_2 = 2 \end{cases} \longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{\vec{b}} \longrightarrow$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}}_{A^T} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}}_{A} = \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$, now

solve $A^TA\vec{x} = \vec{b}$ for $\vec{x}$:

$$\left[\begin{array}{cc|c} 3 & 4 & 3 \\ 4 & 6 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 4 & 3 \\ 1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -2 & -3 \\ 1 & 2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 0 & 2 & 3 \\ 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 3/2 \\ 1 & 0 & -1 \end{array}\right] \Rightarrow$$ L.S.S. is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$$; $$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix} \Rightarrow$$

$$\vec{b} - A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \Rightarrow$$

$$L.S.E. = \|\vec{b} - A\vec{x}\| = \left\| \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \right\|$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4} + 0} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$$

6.) (9 pts.) Assume that A is a 3×4 matrix. Prove that $rank(A) + nullity(A) = 4$.

Since A is 3×4 (4 columns, 3 rows) $\Rightarrow$ the row echelon form of A will have k leading 1's. The solution $\vec{x}$ will have l free variables, and

$$k + l = 4 \text{ for } A\vec{x} = \vec{0}.$$

But $k =$ rank of A and $l =$ nullity of A

$$\Rightarrow \quad rank(A) + nullity(A) = 4$$

7.) (9 pts.) Assume that W is a subspace of a vector space V. Show that $W^{\perp} = \{\vec{v} \in V \mid \vec{v} \cdot \vec{w} = 0 \ for \ all \ \vec{w} \in W\}$ is a subspace of V.

$\vec{0} \in W^{\perp}$ since $\vec{0} \cdot \vec{w} = \vec{0}$ for all $\vec{w} \in W$, so $W^{\perp} \neq \{\}$.

i.) Let $\vec{x}, \vec{v} \in W^{\perp} \Rightarrow \vec{x} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$. Then

$$(\vec{x} + \vec{v}) \cdot \vec{w} = \vec{x} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0 + 0 = 0,$$

so $\vec{x} + \vec{v} \in W^{\perp}$

ii.) Let $\vec{x} \in W^{\perp}$ and k be a scalar. Then $\vec{x} \cdot \vec{w} = 0$ for all $\vec{w} \in W \Rightarrow$

$$(k\vec{x}) \cdot \vec{w} = k(\vec{x} \cdot \vec{w}) = k(0) = 0,$$ so

$k\vec{x} \in W^{\perp}$. Thus $W^{\perp}$ is a subspace.

8.) Let $W = \left\{ \overrightarrow{(a,b,c,d)} \in R^4 \mid a-b=0 \text{ and } c+d=0 \right\}$.

a.) (8 pts.) Show that W is a subspace of R^4. $W \neq \{\ \}$ since

$\overrightarrow{(0,0,0,0)} \in W$ since $0-0=0$, $0+0=0$;

i.) Let $\vec{x} = \overrightarrow{(a_1,b_1,c_1,d_1)}$, $\vec{y} = \overrightarrow{(a_2,b_2,c_2,d_2)} \in W \Rightarrow$

$a_1-b_1=0$, $c_1+d_1=0$ and $a_2-b_2=0$, $c_2+d_2=0$, then

$\vec{x}+\vec{y} = \overrightarrow{(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2)}$ and

$(a_1+a_2)-(b_1+b_2) = (a_1-b_1)+(a_2-b_2) = 0+0=0$,

$(c_1+c_2)+(d_1+d_2) = (c_1+d_1)+(c_2+d_2) = 0+0=0$,

so $\vec{x}+\vec{y} \in W$

ii.) Let $\vec{x} = \overrightarrow{(a,b,c,d)} \in W$ and k a scalar $\Rightarrow$

$k\vec{x} = \overrightarrow{(ka, kb, kc, kd)}$, then

$ka-kb = k(a-b) = k(0) = 0$,

$kc+kd = k(c+d) = k(0) = 0$, so $k\vec{x} \in W$.

Thus, W is a subspace.

b.) (8 pts.) Find a basis for W. $\longrightarrow$ $a=b$, $c=-d$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ b \\ -d \\ d \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$ so basis for W is

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Let $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in W^{\perp} \Rightarrow$

c.) (8 pts.) Find a basis for $W^{\perp}$, the orthogonal complement of W.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x+y=0, \quad \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = -z+w=0 \Rightarrow$$

$x=-y$, $z=w \Rightarrow$

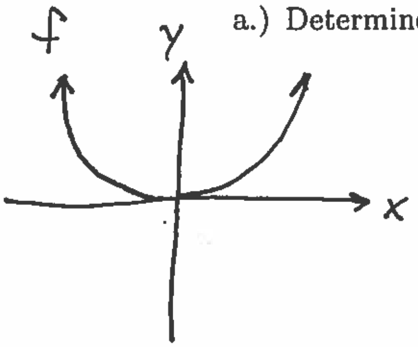
$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -y \\ y \\ w \\ w \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

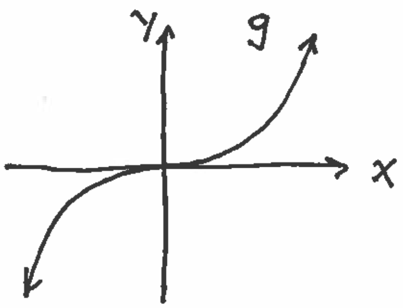
so basis for $W^{\perp}$ is $B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

The following EXTRA CREDIT problem is OPTIONAL. It is worth 10 points.

1.) Let $f(x) = \begin{cases} x^3, \; if \; x \geq 0 \\ -x^3, \; if \; x < 0 \end{cases}$ and $g(x) = x^3$.

a.) Determine if f and g are linearly independent or linearly dependent.





For $x \geq 0$, $f(x) = g(x)$.
For $x < 0$, $f(x) = -g(x)$. Thus
$f(x) \neq k\,g(x)$ for all values
of x, i.e., f and g are
NOT multiples of each
other $\Rightarrow$ f and g are
linearly independent

b.) Compute the Wronskian, $W(x)$, for f and g.

For $x \geq 0$

$$W(x) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 3x^5 - 3x^5 = 0 .$$

For $x < 0$

$$W(x) = \begin{vmatrix} -x^3 & x^3 \\ -3x^2 & 3x^2 \end{vmatrix} = -3x^5 + 3x^5 = 0 .$$

Thus $W(x) = 0$ for all values of x.