where L is a lower triangular matrix with 1's on the main diagonal, D is a diagonal matrix, and U is an upper triangular matrix with 1's on the main diagonal. This is called the ***LDU-decomposition*** (or ***LDU-factorization***) of A.

PLU-Decompositions

Many computer algorithms for solving linear systems perform row interchanges to reduce roundoff error, in which case the existence of an LU-decomposition is not guaranteed. However, it is possible to work around this problem by "preprocessing" the coefficient matrix A so that the row interchanges are performed *prior* to computing the LU-decomposition itself. More specifically, the idea is to create a matrix Q (called a ***permutation matrix***) by multiplying, in sequence, those elementary matrices that produce the row interchanges and then execute them by computing the product QA. This product can then be reduced to row echelon form *without* row interchanges, so it is assured to have an LU-decomposition

$$QA = LU \tag{14}$$

Because the matrix Q is invertible (being a product of elementary matrices), the systems $A\mathbf{x} = \mathbf{b}$ and $QA\mathbf{x} = Q\mathbf{b}$ will have the same solutions. But it follows from (14) that the latter system can be rewritten as $LU\mathbf{x} = Q\mathbf{b}$ and hence can be solved using LU-decomposition.

It is common to see Equation (14) expressed as

$$A = PLU \tag{15}$$

in which $P = Q^{-1}$. This is called a ***PLU-decomposition*** or (***PLU-factorization***) of A.

Exercise Set 9.1

1. Use the method of Example 1 and the LU-decomposition

$$\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

to solve the system

$$\begin{aligned} 3x_1 - 6x_2 &= 0 \\ -2x_1 + 5x_2 &= 1 \end{aligned}$$

2. Use the method of Example 1 and the LU-decomposition

$$\begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

to solve the system

$$\begin{aligned} 3x_1 - 6x_2 - 3x_3 &= -3 \\ 2x_1 \qquad + 6x_3 &= -22 \\ -4x_1 + 7x_2 + 4x_3 &= 3 \end{aligned}$$

► In Exercises 3–6, find an LU-decomposition of the coefficient matrix, and then use the method of Example 1 to solve the system. ◄

3. $\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

4. $\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$

5. $\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

6. $\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$

► In Exercises 7–8, an LU-decomposition of a matrix A is given.

(a) Compute L^{-1} and U^{-1}.

(b) Use the result in part (a) to find the inverse of A. ◄

7. $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix}$;

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}$$

8. The LU-decomposition obtained in Example 2.

9. Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

(a) Find an LU-decomposition of A.

(b) Express A in the form $A = L_1DU_1$, where L_1 is lower triangular with 1's along the main diagonal, U_1 is upper triangular, and D is a diagonal matrix.

(c) Express A in the form $A = L_2U_2$, where L_2 is lower triangular with 1's along the main diagonal and U_2 is upper triangular.

10. (a) Show that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has no LU-decomposition.

(b) Find a PLU-decomposition of this matrix.

In Exercises 11–12, use the given PLU-decomposition of A to solve the linear system $A\mathbf{x} = \mathbf{b}$ by rewriting it as $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$ and solving this system by LU-decomposition.

11. $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$;

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 17 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = PLU$$

12. $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$; $A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 2 & 1 \\ 8 & 1 & 8 \end{bmatrix}$;

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = PLU$$

In Exercises 13–14, find the LDU-decomposition of A.

13. $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & -12 & 6 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix}$

In Exercises 15–16, find a PLU-decomposition of A, and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$ by the method of Exercises 11 and 12.

15. $A = \begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

16. $A = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 1 & 4 \\ 2 & 2 & 5 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$

17. Let $A\mathbf{x} = \mathbf{b}$ be a linear system of n equations in n unknowns, and assume that A is an invertible matrix that can be reduced to row echelon form without row interchanges. How many additions and multiplications are required to solve the system by the method of Example 1?

Working with Proofs

18. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(a) Prove: If $a \neq 0$, then the matrix A has a unique LU-decomposition with 1's along the main diagonal of L.

(b) Find the LU-decomposition described in part (a).

19. Prove: If A is any $n \times n$ matrix, then A can be factored as $A = PLU$, where L is lower triangular, U is upper triangular, and P can be obtained by interchanging the rows of I_n appropriately. [*Hint:* Let U be a row echelon form of A, and let all row interchanges required in the reduction of A to U be performed first.]

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) Every square matrix has an LU-decomposition.

(b) If a square matrix A is row equivalent to an upper triangular matrix U, then A has an LU-decomposition.

(c) If $L_1, L_2, \ldots, L_k$ are $n \times n$ lower triangular matrices, then the product $L_1L_2 \cdots L_k$ is lower triangular.

(d) If an invertible matrix A has an LU-decomposition, then A has a unique LDU-decomposition.

(e) Every square matrix has a PLU-decomposition.

Working with Technology

T1. Technology utilities vary in how they handle LU-decompositions. For example, many utilities perform row interchanges to reduce roundoff error and hence produce PLU-decompositions, even when asked for LU-decompositions. See what happens when you use your utility to find an LU-decomposition of the matrix A in Example 2.

T2. The accompanying figure shows a metal plate whose edges are held at the temperatures shown. It follows from thermodynamic principles that the temperature at each of the six interior nodes will eventually stabilize at a value that is approximately the average of the temperatures at the four neighboring nodes. These are called the ***steady-state temperatures*** at the nodes. Thus, for example, if we denote the steady-state temperatures at the interior nodes in which is called ***column-vector*** form. The choice of notation is often a matter of taste or convenience, but sometimes the nature of a problem will suggest a preferred notation. Notations (15), (16), and (17) will all be used at various places in this text.

Application of Linear Combinations to Color Models

Colors on computer monitors are commonly based on what is called the **RGB *color model***. Colors in this system are created by adding together percentages of the primary colors red (R), green (G), and blue (B). One way to do this is to identify the primary colors with the vectors

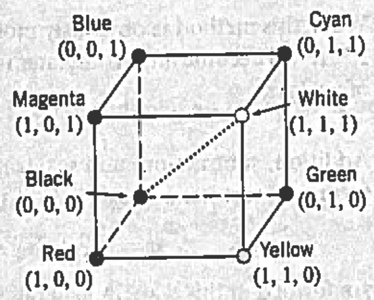
$$\mathbf{r} = (1, 0, 0) \quad \text{(pure red)},$$
$$\mathbf{g} = (0, 1, 0) \quad \text{(pure green)},$$
$$\mathbf{b} = (0, 0, 1) \quad \text{(pure blue)}$$

in R^3 and to create all other colors by forming linear combinations of $\mathbf{r}$, $\mathbf{g}$, and $\mathbf{b}$ using coefficients between 0 and 1, inclusive; these coefficients represent the percentage of each pure color in the mix.

The set of all such color vectors is called **RGB *space*** or the **RGB *color cube*** (Figure 3.1.14). Thus, each color vector $\mathbf{c}$ in this cube is expressible as a linear combination of the form

$$\begin{aligned}\mathbf{c} &= k_1\mathbf{r} + k_2\mathbf{g} + k_3\mathbf{b} \\ &= k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) \\ &= (k_1, k_2, k_3)\end{aligned}$$

where $0 \le k_i \le 1$. As indicated in the figure, the corners of the cube represent the pure primary colors together with the colors black, white, magenta, cyan, and yellow. The vectors along the diagonal running from black to white correspond to shades of gray.

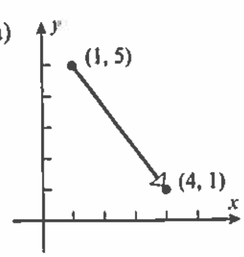


▶ Figure 3.1.14

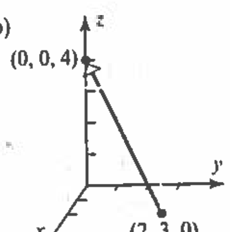
Exercise Set 3.1

▶ In Exercises 1–2, find the components of the vector. ◀

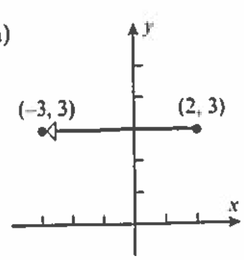
1. (a)



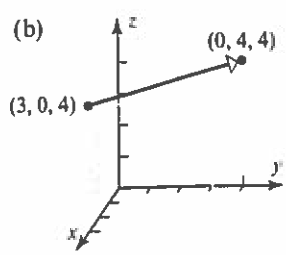
(b)



2. (a)



(b)



▶ In Exercises 3–4, find the components of the vector $\overrightarrow{P_1P_2}$. ◀

3. (a) $P_1(3, 5)$, $P_2(2, 8)$ (b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$

4. (a) $P_1(-6, 2)$, $P_2(-4, -1)$ (b) $P_1(0, 0, 0)$, $P_2(-1, 6, 1)$

5. (a) Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1, 2)$ and whose initial point is $A(1, 1)$.
   (b) Find the initial point of the vector that is equivalent to $\mathbf{u} = (1, 1, 3)$ and whose terminal point is $B(-1, -1, 2)$.

6. (a) Find the initial point of the vector that is equivalent to $\mathbf{u} = (1, 2)$ and whose terminal point is $B(2, 0)$.
   (b) Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1, 1, 3)$ and whose initial point is $A(0, 2, 0)$.

7. Find an initial point P of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with terminal point $Q(3, 0, -5)$ and such that
   (a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (4, -2, -1)$.
   (b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (4, -2, -1)$.

8. Find a terminal point Q of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with initial point $P(-1, 3, -5)$ and such that
   (a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (6, 7, -3)$.
   (b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (6, 7, -3)$.

9. Let $\mathbf{u} = (4, -1)$, $\mathbf{v} = (0, 5)$, and $\mathbf{w} = (-3, -3)$. Find the components of
   (a) $\mathbf{u} + \mathbf{w}$ (b) $\mathbf{v} - 3\mathbf{u}$
   (c) $2(\mathbf{u} - 5\mathbf{w})$ (d) $3\mathbf{v} - 2(\mathbf{u} + 2\mathbf{w})$

10. Let $\mathbf{u} = (-3, 1, 2)$, $\mathbf{v} = (4, 0, -8)$, and $\mathbf{w} = (6, -1, -4)$. Find the components of
   (a) $\mathbf{v} - \mathbf{w}$ (b) $6\mathbf{u} + 2\mathbf{v}$
   (c) $-3(\mathbf{v} - 8\mathbf{w})$ (d) $(2\mathbf{u} - 7\mathbf{w}) - (8\mathbf{v} + \mathbf{u})$

11. Let $\mathbf{u} = (-3, 2, 1, 0)$, $\mathbf{v} = (4, 7, -3, 2)$, and $\mathbf{w} = (5, -2, 8, 1)$. Find the components of
   (a) $\mathbf{v} - \mathbf{w}$ (b) $-\mathbf{u} + (\mathbf{v} - 4\mathbf{w})$
   (c) $6(\mathbf{u} - 3\mathbf{v})$ (d) $(6\mathbf{v} - \mathbf{w}) - (4\mathbf{u} + \mathbf{v})$

12. Let $\mathbf{u} = (1, 2, -3, 5, 0)$, $\mathbf{v} = (0, 4, -1, 1, 2)$, and $\mathbf{w} = (7, 1, -4, -2, 3)$. Find the components of
   (a) $\mathbf{v} + \mathbf{w}$ (b) $3(2\mathbf{u} - \mathbf{v})$
   (c) $(3\mathbf{u} - \mathbf{v}) - (2\mathbf{u} + 4\mathbf{w})$ (d) $\frac{1}{2}(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v}$

13. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the vectors in Exercise 11. Find the components of the vector $\mathbf{x}$ that satisfies the equation $3\mathbf{u} + \mathbf{v} - 2\mathbf{w} = 3\mathbf{x} + 2\mathbf{w}$.

14. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the vectors in Exercise 12. Find the components of the vector $\mathbf{x}$ that satisfies the equation $2\mathbf{u} - \mathbf{v} + \mathbf{x} = 7\mathbf{x} + \mathbf{w}$.

15. Which of the following vectors in R^6, if any, are parallel to $\mathbf{u} = (-2, 1, 0, 3, 5, 1)$?
   (a) $(4, 2, 0, 6, 10, 2)$
   (b) $(4, -2, 0, -6, -10, -2)$
   (c) $(0, 0, 0, 0, 0, 0)$

16. For what value(s) of t, if any, is the given vector parallel to $\mathbf{u} = (4, -1)$?
   (a) $(8t, -2)$ (b) $(8t, 2t)$ (c) $(1, t^2)$

17. Let $\mathbf{u} = (1, -1, 3, 5)$ and $\mathbf{v} = (2, 1, 0, -3)$. Find scalars a and b so that $a\mathbf{u} + b\mathbf{v} = (1, -4, 9, 18)$.

18. Let $\mathbf{u} = (2, 1, 0, 1, -1)$ and $\mathbf{v} = (-2, 3, 1, 0, 2)$. Find scalars a and b so that $a\mathbf{u} + b\mathbf{v} = (-8, 8, 3, -1, 7)$.

In Exercises 19–20, find scalars c_1, c_2, and c_3 for which the equation is satisfied.

19. $c_1(1, -1, 0) + c_2(3, 2, 1) + c_3(0, 1, 4) = (-1, 1, 19)$

20. $c_1(-1, 0, 2) + c_2(2, 2, -2) + c_3(1, -2, 1) = (-6, 12, 4)$

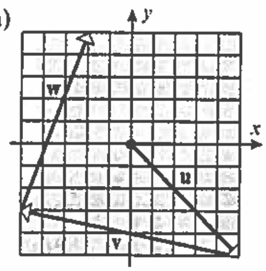
21. Show that there do not exist scalars c_1, c_2, and c_3 such that
$$c_1(-2, 9, 6) + c_2(-3, 2, 1) + c_3(1, 7, 5) = (0, 5, 4)$$

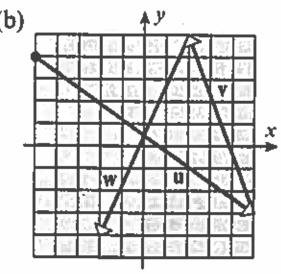
22. Show that there do not exist scalars c_1, c_2, and c_3 such that
$$c_1(1, 0, 1, 0) + c_2(1, 0, -2, 1) + c_3(2, 0, 1, 2) = (1, -2, 2, 3)$$

23. Let P be the point $(2, 3, -2)$ and Q the point $(7, -4, 1)$.
   (a) Find the midpoint of the line segment connecting the points P and Q.
   (b) Find the point on the line segment connecting the points P and Q that is $\frac{3}{4}$ of the way from P to Q.

24. In relation to the points P_1 and P_2 in Figure 3.1.12, what can you say about the terminal point of the following vector if its initial point is at the origin?
$$\mathbf{u} = \overrightarrow{OP_1} + \tfrac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})$$

25. In each part, find the components of the vector $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

(a)


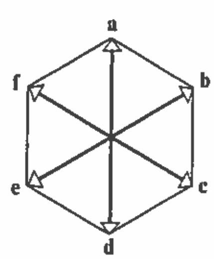
(b)


26. Referring to the vectors pictured in Exercise 25, find the components of the vector $\mathbf{u} - \mathbf{v} + \mathbf{w}$.

27. Let P be the point $(1, 3, 7)$. If the point $(4, 0, -6)$ is the midpoint of the line segment connecting P and Q, what is Q?

28. If the sum of three vectors in R^3 is zero, must they lie in the same plane? Explain.

29. Consider the regular hexagon shown in the accompanying figure.
   (a) What is the sum of the six radial vectors that run from the center to the vertices?
   (b) How is the sum affected if each radial vector is multiplied by $\frac{1}{2}$?
   (c) What is the sum of the five radial vectors that remain if $\mathbf{a}$ is removed?
   (d) Discuss some variations and generalizations of the result in part (c).



◀ Figure Ex-29

30. What is the sum of all radial vectors of a regular n-sided polygon? (See Exercise 29.)

Working with Proofs

31. Prove parts (*a*), (*c*), and (*d*) of Theorem 3.1.1.

32. Prove parts (*e*)–(*h*) of Theorem 3.1.1.

33. Prove parts (*a*)–(*c*) of Theorem 3.1.2.

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two equivalent vectors must have the same initial point.

(b) The vectors (a, b) and $(a, b, 0)$ are equivalent.

(c) If k is a scalar and $\mathbf{v}$ is a vector, then $\mathbf{v}$ and $k\mathbf{v}$ are parallel if and only if $k \geq 0$.

(d) The vectors $\mathbf{v} + (\mathbf{u} + \mathbf{w})$ and $(\mathbf{w} + \mathbf{v}) + \mathbf{u}$ are the same.

(e) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(f) If a and b are scalars such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors.

(g) Collinear vectors with the same length are equal.

(h) If $(a, b, c) + (x, y, z) = (x, y, z)$, then (a, b, c) must be the zero vector.

(i) If k and m are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors, then

$$(k + m)(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + m\mathbf{v}$$

(j) If the vectors $\mathbf{v}$ and $\mathbf{w}$ are given, then the vector equation

$$3(2\mathbf{v} - \mathbf{x}) = 5\mathbf{x} - 4\mathbf{w} + \mathbf{v}$$

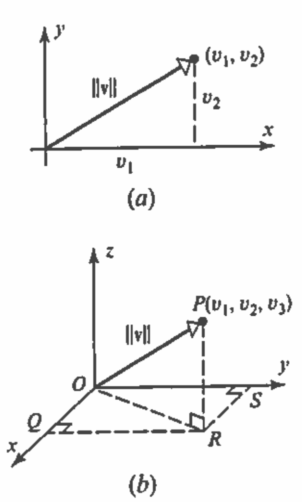
can be solved for $\mathbf{x}$.

(k) The linear combinations $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ and $b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ can only be equal if $a_1 = b_1$ and $a_2 = b_2$.

3.2 Norm, Dot Product, and Distance in R^n

In this section we will be concerned with the notions of length and distance as they relate to vectors. We will first discuss these ideas in R^2 and R^3 and then extend them algebraically to R^n.

Norm of a Vector



▲ Figure 3.2.1

In this text we will denote the length of a vector $\mathbf{v}$ by the symbol $\|\mathbf{v}\|$, which is read as the *norm* of $\mathbf{v}$, the *length* of $\mathbf{v}$, or the *magnitude* of $\mathbf{v}$ (the term "norm" being a common mathematical synonym for length). As suggested in Figure 3.2.1*a*, it follows from the Theorem of Pythagoras that the norm of a vector (v_1, v_2) in R^2 is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \tag{1}$$

Similarly, for a vector (v_1, v_2, v_3) in R^3, it follows from Figure 3.2.1*b* and two applications of the Theorem of Pythagoras that

$$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (QR)^2 + (RP)^2 = v_1^2 + v_2^2 + v_3^2$$

and hence that

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{2}$$

Motivated by the pattern of Formulas (1) and (2), we make the following definition.

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

Thus, if the row vectors of A are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column vectors of B are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix} \tag{28}$$

Exercise Set 3.2

► In Exercises 1–2, find the norm of $\mathbf{v}$, and a unit vector that is oppositely directed to $\mathbf{v}$. ◄

1. (a) $\mathbf{v} = (2, 2, 2)$ (b) $\mathbf{v} = (1, 0, 2, 1, 3)$

2. (a) $\mathbf{v} = (1, -1, 2)$ (b) $\mathbf{v} = (-2, 3, 3, -1)$

► In Exercises 3–4, evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$. ◄

3. (a) $\|\mathbf{u} + \mathbf{v}\|$ (b) $\|\mathbf{u}\| + \|\mathbf{v}\|$
   (c) $\|-2\mathbf{u} + 2\mathbf{v}\|$ (d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$

4. (a) $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$ (b) $\|\mathbf{u} - \mathbf{v}\|$
   (c) $\|3\mathbf{v}\| - 3\|\mathbf{v}\|$ (d) $\|\mathbf{u}\| - \|\mathbf{v}\|$

► In Exercises 5–6, evaluate the given expression with $\mathbf{u} = (-2, -1, 4, 5)$, $\mathbf{v} = (3, 1, -5, 7)$, and $\mathbf{w} = (-6, 2, 1, 1)$. ◄

5. (a) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$ (b) $\|3\mathbf{u}\| - 5\|\mathbf{v}\| + \|\mathbf{w}\|$
   (c) $\|-\|\mathbf{u}\|\mathbf{v}\|$

6. (a) $\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$ (b) $\|\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\|$

7. Let $\mathbf{v} = (-2, 3, 0, 6)$. Find all scalars k such that $\|k\mathbf{v}\| = 5$.

8. Let $\mathbf{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|k\mathbf{v}\| = 4$.

► In Exercises 9–10, find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$. ◄

9. (a) $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (2, 2, -4)$
   (b) $\mathbf{u} = (1, 1, 4, 6)$, $\mathbf{v} = (2, -2, 3, -2)$

10. (a) $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$
    (b) $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

► In Exercises 11–12, find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°. ◄

11. (a) $\mathbf{u} = (3, 3, 3)$, $\mathbf{v} = (1, 0, 4)$
    (b) $\mathbf{u} = (0, -2, -1, 1)$, $\mathbf{v} = (-3, 2, 4, 4)$

12. (a) $\mathbf{u} = (1, 2, -3, 0)$, $\mathbf{v} = (5, 1, 2, -2)$
    (b) $\mathbf{u} = (0, 1, 1, 1, 2)$, $\mathbf{v} = (2, 1, 0, -1, 3)$

13. Suppose that a vector $\mathbf{a}$ in the xy-plane has a length of 9 units and points in a direction that is 120° counterclockwise from the positive x-axis, and a vector $\mathbf{b}$ in that plane has a length of 5 units and points in the positive y-direction. Find $\mathbf{a} \cdot \mathbf{b}$.

14. Suppose that a vector $\mathbf{a}$ in the xy-plane points in a direction that is 47° counterclockwise from the positive x-axis, and a vector $\mathbf{b}$ in that plane points in a direction that is 43° clockwise from the positive x-axis. What can you say about the value of $\mathbf{a} \cdot \mathbf{b}$?

► In Exercises 15–16, determine whether the expression makes sense mathematically. If not, explain why. ◄

15. (a) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
    (c) $\|\mathbf{u} \cdot \mathbf{v}\|$ (d) $(\mathbf{u} \cdot \mathbf{v}) - \|\mathbf{u}\|$

16. (a) $\|\mathbf{u}\| \cdot \|\mathbf{v}\|$ (b) $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{w}$
    (c) $(\mathbf{u} \cdot \mathbf{v}) - k$ (d) $k \cdot \mathbf{u}$

► In Exercises 17–18, verify that the Cauchy–Schwarz inequality holds. ◄

17. (a) $\mathbf{u} = (-3, 1, 0)$, $\mathbf{v} = (2, -1, 3)$
    (b) $\mathbf{u} = (0, 2, 2, 1)$, $\mathbf{v} = (1, 1, 1, 1)$

18. (a) $\mathbf{u} = (4, 1, 1)$, $\mathbf{v} = (1, 2, 3)$
    (b) $\mathbf{u} = (1, 2, 1, 2, 3)$, $\mathbf{v} = (0, 1, 1, 5, -2)$

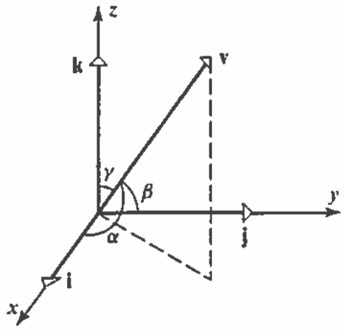
19. Let $\mathbf{r}_0 = (x_0, y_0)$ be a fixed vector in R^2. In each part, describe in words the set of all vectors $\mathbf{r} = (x, y)$ that satisfy the stated condition.
    (a) $\|\mathbf{r} - \mathbf{r}_0\| = 1$ (b) $\|\mathbf{r} - \mathbf{r}_0\| \le 1$ (c) $\|\mathbf{r} - \mathbf{r}_0\| > 1$

20. Repeat the directions of Exercise 19 for vectors $\mathbf{r} = (x, y, z)$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$ in R^3.

► Exercises 21–25 The direction of a nonzero vector $\mathbf{v}$ in an xyz-coordinate system is completely determined by the angles α, β, and γ between $\mathbf{v}$ and the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Figure Ex-21). These are called the ***direction angles*** of $\mathbf{v}$, and their cosines are called the ***direction cosines*** of $\mathbf{v}$. ◄

21. Use Formula (13) to show that the direction cosines of a vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 are

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$



◀ Figure Ex-21

22. Use the result in Exercise 21 to show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

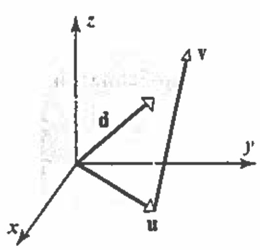
23. Show that two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in R^3 are orthogonal if and only if their direction cosines satisfy

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$$

24. The accompanying figure shows a cube.

(a) Find the angle between the vectors $\mathbf{d}$ and $\mathbf{u}$ to the nearest degree.

(b) Make a conjecture about the angle between the vectors $\mathbf{d}$ and $\mathbf{v}$, and confirm your conjecture by computing the angle.



◀ Figure Ex-24

25. Estimate, to the nearest degree, the angles that a diagonal of a box with dimensions 10 cm × 15 cm × 25 cm makes with the edges of the box.

26. If $\|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$, what are the largest and smallest values possible for $\|\mathbf{v} - \mathbf{w}\|$? Give a geometric explanation of your results.

27. What can you say about two nonzero vectors, $\mathbf{u}$ and $\mathbf{v}$, that satisfy the equation $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$?

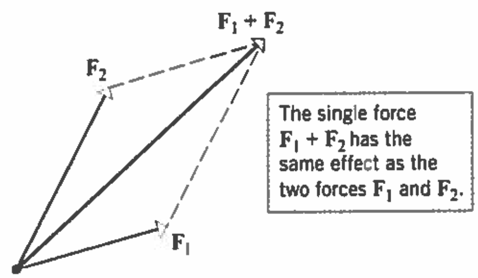
28. (a) What relationship must hold for the point $\mathbf{p} = (a, b, c)$ to be equidistant from the origin and the xz-plane? Make sure that the relationship you state is valid for positive and negative values of a, b, and c.

(b) What relationship must hold for the point $\mathbf{p} = (a, b, c)$ to be farther from the origin than from the xz-plane? Make sure that the relationship you state is valid for positive and negative values of a, b, and c.

29. State a procedure for finding a vector of a specified length m that points in the same direction as a given vector $\mathbf{v}$.

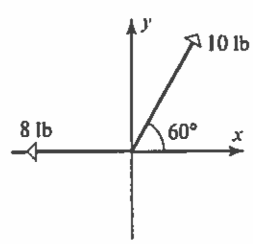
30. Under what conditions will the triangle inequality (Theorem 3.2.5a) be an equality? Explain your answer geometrically.

► Exercises 31–32 The effect that a force has on an object depends on the magnitude of the force and the direction in which it is applied. Thus, forces can be regarded as vectors and represented as arrows in which the length of the arrow specifies the magnitude of the force, and the direction of the arrow specifies the direction in which the force is applied. It is a fact of physics that force vectors obey the parallelogram law in the sense that if two force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied at a point on an object, then the effect is the same as if the single force $\mathbf{F}_1 + \mathbf{F}_2$ (called the ***resultant***) were applied at that point (see accompanying figure). Forces are commonly measured in units called pounds-force (abbreviated lbf) or Newtons (abbreviated N). ◄

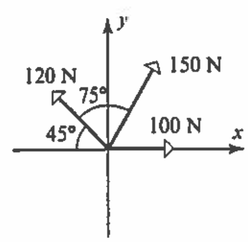


31. A particle is said to be in ***static equilibrium*** if the resultant of all forces applied to it is zero. For the forces in the accompanying figure, find the resultant $\mathbf{F}$ that must be applied to the indicated point to produce static equilibrium. Describe $\mathbf{F}$ by giving its magnitude and the angle in degrees that it makes with the positive x-axis.

32. Follow the directions of Exercise 31.



▲ Figure Ex-31



▲ Figure Ex-32

Working with Proofs

33. Prove parts (a) and (b) of Theorem 3.2.1.

34. Prove parts (a) and (c) of Theorem 3.2.3.

35. Prove parts (d) and (e) of Theorem 3.2.3.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) If each component of a vector in R^3 is doubled, the norm of that vector is doubled.

(b) In R^2, the vectors of norm 5 whose initial points are at the origin have terminal points lying on a circle of radius 5 centered at the origin.

(c) Every vector in R^n has a positive norm.

(d) If $\mathbf{v}$ is a nonzero vector in R^n, there are exactly two unit vectors that are parallel to $\mathbf{v}$.

(e) If $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 1$, and $\mathbf{u}\cdot\mathbf{v} = 1$, then the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$ radians.

(f) The expressions $(\mathbf{u}\cdot\mathbf{v}) + \mathbf{w}$ and $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ are both meaningful and equal to each other.

(g) If $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(h) If $\mathbf{u}\cdot\mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

(i) In R^2, if $\mathbf{u}$ lies in the first quadrant and $\mathbf{v}$ lies in the third quadrant, then $\mathbf{u}\cdot\mathbf{v}$ cannot be positive.

(j) For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in R^n, we have

$$\|\mathbf{u}+\mathbf{v}+\mathbf{w}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\|$$

Working with Technology

T1. Let $\mathbf{u}$ be a vector in R^{100} whose ith component is i, and let $\mathbf{v}$ be the vector in R^{100} whose ith component is $1/(i+1)$. Find the dot product of $\mathbf{u}$ and $\mathbf{v}$.

T2. Find, to the nearest degree, the angles that a diagonal of a box with dimensions 10 cm $\times$ 11 cm $\times$ 25 cm makes with the edges of the box.

3.3 Orthogonality

In the last section we defined the notion of "angle" between vectors in R^n. In this section we will focus on the notion of "perpendicularity." Perpendicular vectors in R^n play an important role in a wide variety of applications.

Orthogonal Vectors

Recall from Formula (20) in the previous section that the angle θ between two *nonzero* vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is defined by the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

It follows from this that $\theta = \pi/2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$. Thus, we make the following definition.

DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u}\cdot\mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

▶ EXAMPLE 1 Orthogonal Vectors

(a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4.

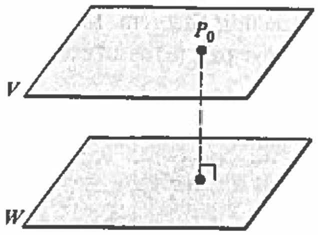
(b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3. Show that each ordered pair of vectors in S is orthogonal.

Solution (a) The vectors are orthogonal since

$$\mathbf{u}\cdot\mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Solution (b) It suffices to show that

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$$



▲ Figure 3.3.7 The distance between the parallel planes V and W is equal to the distance between P_0 and W.

The third distance problem posed above is to find the distance between two parallel planes in R^3. As suggested in Figure 3.3.7, the distance between a plane V and a plane W can be obtained by finding any point P_0 in one of the planes, and computing the distance between that point and the other plane. Here is an example.

▶ EXAMPLE 8 Distance Between Parallel Planes

The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $(1, 2, -2)$ and $(2, 4, -4)$, are parallel vectors. Find the distance between these planes.

Solution To find the distance D between the planes, we can select an arbitrary point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (16), the distance between P_0 and the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|2(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6} \blacktriangleleft$$

Exercise Set 3.3

In Exercises 1–2, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors.

1. (a) $\mathbf{u} = (6, 1, 4)$, $\mathbf{v} = (2, 0, -3)$
   (b) $\mathbf{u} = (0, 0, -1)$, $\mathbf{v} = (1, 1, 1)$
   (c) $\mathbf{u} = (3, -2, 1, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$
   (d) $\mathbf{u} = (5, -4, 0, 3)$, $\mathbf{v} = (-4, 1, -3, 7)$

2. (a) $\mathbf{u} = (2, 3)$, $\mathbf{v} = (5, -7)$
   (b) $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (0, 0, 0)$
   (c) $\mathbf{u} = (1, -5, 4)$, $\mathbf{v} = (3, 3, 3)$
   (d) $\mathbf{u} = (4, 1, -2, 5)$, $\mathbf{v} = (-1, 5, 3, 1)$

In Exercises 3–6, find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal.

3. $P(-1, 3, -2)$; $\mathbf{n} = (-2, 1, -1)$

4. $P(1, 1, 4)$; $\mathbf{n} = (1, 9, 8)$

5. $P(2, 0, 0)$; $\mathbf{n} = (0, 0, 2)$

6. $P(0, 0, 0)$; $\mathbf{n} = (1, 2, 3)$

In Exercises 7–10, determine whether the given planes are parallel.

7. $4x - y + 2z = 5$ and $7x - 3y + 4z = 8$

8. $x - 4y - 3z - 2 = 0$ and $3x - 12y - 9z - 7 = 0$

9. $2y = 8x - 4z + 5$ and $x = \frac{1}{2}z + \frac{1}{4}y$

10. $(-4, 1, 2) \cdot (x, y, z) = 0$ and $(8, -2, -4) \cdot (x, y, z) = 0$

In Exercises 11–12, determine whether the given planes are perpendicular.

11. $3x - y + z - 4 = 0$, $x + 2z = -1$

12. $x - 2y + 3z = 4$, $-2x + 5y + 4z = -1$

In Exercises 13–14, find $\|\text{proj}_{\mathbf{a}}\mathbf{u}\|$.

13. (a) $\mathbf{u} = (1, -2)$, $\mathbf{a} = (-4, -3)$
    (b) $\mathbf{u} = (3, 0, 4)$, $\mathbf{a} = (2, 3, 3)$

14. (a) $\mathbf{u} = (5, 6)$, $\mathbf{a} = (2, -1)$
    (b) $\mathbf{u} = (3, -2, 6)$, $\mathbf{a} = (1, 2, -7)$

In Exercises 15–20, find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

15. $\mathbf{u} = (6, 2)$, $\mathbf{a} = (3, -9)$

16. $\mathbf{u} = (-1, -2)$, $\mathbf{a} = (-2, 3)$

17. $\mathbf{u} = (3, 1, -7)$, $\mathbf{a} = (1, 0, 5)$

18. $\mathbf{u} = (2, 0, 1)$, $\mathbf{a} = (1, 2, 3)$

19. $\mathbf{u} = (2, 1, 1, 2)$, $\mathbf{a} = (4, -4, 2, -2)$

20. $\mathbf{u} = (5, 0, -3, 7)$, $\mathbf{a} = (2, 1, -1, -1)$

In Exercises 21–24, find the distance between the point and the line.

21. $(-3, 1)$; $4x + 3y + 4 = 0$

22. $(-1, 4)$; $x - 3y + 2 = 0$

23. $(2, -5)$; $y = -4x + 2$

24. $(1, 8)$; $3x + y = 5$

In Exercises 25–26, find the distance between the point and the plane.

25. $(3, 1, -2)$; $x + 2y - 2z = 4$

26. $(-1, -1, 2)$; $2x + 5y - 6z = 4$

In Exercises 27–28, find the distance between the given parallel planes.

27. $2x - y - z = 5$ and $-4x + 2y + 2z = 12$

28. $2x - y + z = 1$ and $2x - y + z = -1$

29. Find a unit vector that is orthogonal to both $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (0, 1, 1)$.

30. (a) Show that $\mathbf{v} = (a, b)$ and $\mathbf{w} = (-b, a)$ are orthogonal vectors.

(b) Use the result in part (a) to find two vectors that are orthogonal to $\mathbf{v} = (2, -3)$.

(c) Find two unit vectors that are orthogonal to $\mathbf{v} = (-3, 4)$.

31. Do the points $A(1, 1, 1)$, $B(-2, 0, 3)$, and $C(-3, -1, 1)$ form the vertices of a right triangle? Explain.

32. Repeat Exercise 31 for the points $A(3, 0, 2)$, $B(4, 3, 0)$, and $C(8, 1, -1)$.

33. Show that if $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is orthogonal to $k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for all scalars k_1 and k_2.

34. Is it possible to have $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a}$? Explain.

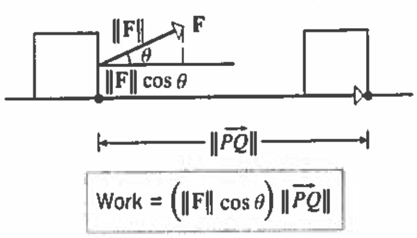
Exercises 35–37 In physics and engineering the ***work*** W performed by a ***constant force*** $\mathbf{F}$ applied in the ***direction of motion*** to an object moving a distance d on a straight line is defined to be

$$W = \|\mathbf{F}\|d \quad \text{(force magnitude times distance)}$$

In the case where the applied force is constant but makes an angle θ with the direction of motion, and where the object moves along a line from a point P to a point Q, we call $\overrightarrow{PQ}$ the ***displacement*** and define the work performed by the force to be

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\|\|\overrightarrow{PQ}\| \cos\theta$$

(see accompanying figure). Common units of work are ft-lb (foot pounds) or Nm (Newton meters).

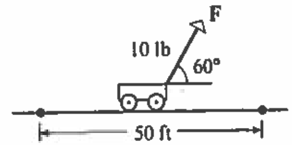


35. Show that the work performed by a constant force (not necessarily in the direction of motion) can be expressed as

$$W = \pm\|\overrightarrow{PQ}\|\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|$$

and explain when the + sign should be used and when the − sign should be used.

36. As illustrated in the accompanying figure, a wagon is pulled horizontally by exerting a force of 10 lb on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon 50 ft?



37. A sailboat travels 100 m due north while the wind exerts a force of 500 N toward the northeast. How much work does the wind do?

Working with Proofs

38. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in 2- or 3-space, and let $k = \|\mathbf{u}\|$ and $l = \|\mathbf{v}\|$. Prove that the vector $\mathbf{w} = l\mathbf{u} + k\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

39. Prove part (*a*) of Theorem 3.3.4.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) The vectors $(3, -1, 2)$ and $(0, 0, 0)$ are orthogonal.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then for all nonzero scalars k and m, $k\mathbf{u}$ and $m\mathbf{v}$ are orthogonal vectors.

(c) The orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$ is perpendicular to the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

(d) If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors, then for every nonzero vector $\mathbf{u}$, we have

$$\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{b}}(\mathbf{u})) = \mathbf{0}$$

(e) If $\mathbf{a}$ and $\mathbf{u}$ are nonzero vectors, then

$$\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{a}}(\mathbf{u})) = \text{proj}_{\mathbf{a}}(\mathbf{u})$$

(f) If the relationship

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{a}}\mathbf{v}$$

holds for some nonzero vector $\mathbf{a}$, then $\mathbf{u} = \mathbf{v}$.

(g) For all vectors $\mathbf{u}$ and $\mathbf{v}$, it is true that

$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$$

Working with Technology

T1. Find the lengths of the sides and the interior angles of the triangle in R^4 whose vertices are

$$P(2, 4, 2, 4, 2), \quad Q(6, 4, 4, 4, 6), \quad R(5, 7, 5, 7, 2)$$

T2. Express the vector $\mathbf{u} = (2, 3, 1, 2)$ in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a} = (-1, 0, 2, 1)$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

Exercise Set 2.1

In Exercises 1–2, find all the minors and cofactors of the matrix A.

1. $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$ 2. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix}$

3. Let
$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$
Find
(a) M_{13} and C_{13}. (b) M_{23} and C_{23}.
(c) M_{22} and C_{22}. (d) M_{21} and C_{21}.

4. Let
$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$
Find
(a) M_{32} and C_{32}. (b) M_{44} and C_{44}.
(c) M_{41} and C_{41}. (d) M_{24} and C_{24}.

In Exercises 5–8, evaluate the determinant of the given matrix. If the matrix is invertible, use Equation (2) to find its inverse.

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ 6. $\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$ 7. $\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$ 8. $\begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$

In Exercises 9–14, use the arrow technique to evaluate the determinant.

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$ 10. $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$ 12. $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix}$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$ 14. $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

In Exercises 15–18, find all values of λ for which $\det(A) = 0$.

15. $A = \begin{bmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{bmatrix}$ 16. $A = \begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{bmatrix}$

17. $A = \begin{bmatrix} \lambda-1 & 0 \\ 2 & \lambda+1 \end{bmatrix}$ 18. $A = \begin{bmatrix} \lambda-4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-5 \end{bmatrix}$

19. Evaluate the determinant in Exercise 13 by a cofactor expansion along
(a) the first row. (b) the first column.
(c) the second row. (d) the second column.
(e) the third row. (f) the third column.

20. Evaluate the determinant in Exercise 12 by a cofactor expansion along
(a) the first row. (b) the first column.
(c) the second row. (d) the second column.
(e) the third row. (f) the third column.

In Exercises 21–26, evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice.

21. $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$ 22. $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$ 24. $A = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$

25. $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

In Exercises 27–32, evaluate the determinant of the given matrix by inspection.

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 28. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{bmatrix}$ 30. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ 32. $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{bmatrix}$

33. In each part, show that the value of the determinant is independent of θ.

(a) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

(b) $\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$

34. Show that the matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

35. By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix} \quad \text{and} \quad d_2 = \begin{vmatrix} a+\lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

36. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

37. What can you say about an nth-order determinant all of whose entries are 1? Explain.

38. What is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant? Explain.

39. Explain why the determinant of a matrix with integer entries must be an integer.

Working with Proofs

40. Prove that (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear points if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

41. Prove that the equation of the line through the distinct points (a_1, b_1) and (a_2, b_2) can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

42. Prove that if A is upper triangular and B_{ij} is the matrix that results when the ith row and jth column of A are deleted, then B_{ij} is upper triangular if $i < j$.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$.

(b) Two square matrices that have the same determinant must have the same size.

(c) The minor M_{ij} is the same as the cofactor C_{ij} if $i + j$ is even.

(d) If A is a 3×3 symmetric matrix, then $C_{ij} = C_{ji}$ for all i and j.

(e) The number obtained by a cofactor expansion of a matrix A is independent of the row or column chosen for the expansion.

(f) If A is a square matrix whose minors are all zero, then $\det(A) = 0$.

(g) The determinant of a lower triangular matrix is the sum of the entries along the main diagonal.

(h) For every square matrix A and every scalar c, it is true that $\det(cA) = c\det(A)$.

(i) For all square matrices A and B, it is true that

$$\det(A + B) = \det(A) + \det(B)$$

(j) For every 2×2 matrix A it is true that $\det(A^2) = (\det(A))^2$.

Working with Technology

T1. (a) Use the determinant capability of your technology utility to find the determinant of the matrix

$$A = \begin{bmatrix} 4.2 & -1.3 & 1.1 & 6.0 \\ 0.0 & 0.0 & -3.2 & 3.4 \\ 4.5 & 1.3 & 0.0 & 14.8 \\ 4.7 & 1.0 & 3.4 & 2.3 \end{bmatrix}$$

(b) Compare the result obtained in part (a) to that obtained by a cofactor expansion along the second row of A.

T2. Let A^n be the $n \times n$ matrix with 2's along the main diagonal, 1's along the diagonal lines immediately above and below the main diagonal, and zeros everywhere else. Make a conjecture about the relationship between n and $\det(A_n)$.

Exercise Set 2.2

In Exercises 1–4, verify that $\det(A) = \det(A^T)$.

1. $A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} -6 & 1 \\ 2 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & 2 & -1 \\ 0 & 2 & -3 \\ -1 & 1 & 5 \end{bmatrix}$

In Exercises 5–8, find the determinant of the given elementary matrix by inspection.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 9–14, evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

9. $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

In Exercises 15–22, evaluate the determinant, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

15. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

16. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$

17. $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

18. $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$

19. $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$

21. $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

22. $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$

23. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

24. Verify the formulas in parts (a) and (b) and then make a conjecture about a general result of which these results are special cases.

(a) $\det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -a_{13}a_{22}a_{31}$

(b) $\det\begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$

In Exercises 25–28, confirm the identities without evaluating the determinants directly.

25. $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

26. $\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

27. $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

28. $\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Exercise Set 4.1

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

   (a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$.

   (b) In words, explain why V is closed under addition and scalar multiplication.

   (c) Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

   (d) Show that Axioms 7, 8, and 9 hold.

   (e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

2. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

   (a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (0, 4)$, $\mathbf{v} = (1, -3)$, and $k = 2$.

   (b) Show that $(0, 0) \neq \mathbf{0}$.

   (c) Show that $(-1, -1) = \mathbf{0}$.

   (d) Show that Axiom 5 holds by producing an ordered pair $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for $\mathbf{u} = (u_1, u_2)$.

   (e) Find two vector space axioms that fail to hold.

In Exercises 3–12, determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail.

3. The set of all real numbers with the standard operations of addition and multiplication.

4. The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

5. The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2.

6. The set of all n-tuples of real numbers that have the form $(x, x, \dots, x)$ with the standard operations on R^n.

7. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

8. The set of all 2×2 invertible matrices with the standard matrix addition and scalar multiplication.

9. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

10. The set of all real-valued functions f defined everywhere on the real line and such that $f(1) = 0$ with the operations used in Example 6.

11. The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

12. The set of polynomials of the form $a_0 + a_1x$ with the operations

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x$$

and

$$k(a_0 + a_1x) = (ka_0) + (ka_1)x$$

13. Verify Axioms 3, 7, 8, and 9 for the vector space given in Example 4.

14. Verify Axioms 1, 2, 3, 7, 8, 9, and 10 for the vector space given in Example 6.

15. With the addition and scalar multiplication operations defined in Example 7, show that $V = R^2$ satisfies Axioms 1–9.

16. Verify Axioms 1, 2, 3, 6, 8, 9, and 10 for the vector space given in Example 8.

17. Show that the set of all points in R^2 lying on a line is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the line passes through the origin.

18. Show that the set of all points in R^3 lying in a plane is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the plane passes through the origin.

In Exercises 19–20, let V be the vector space of positive real numbers with the vector space operations given in Example 8. Let $\mathbf{u} = u$ be any vector in V, and rewrite the vector statement as a statement about real numbers.

19. $-\mathbf{u} = (-1)\mathbf{u}$

20. $k\mathbf{u} = \mathbf{0}$ if and only if $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Working with Proofs

21. The argument that follows proves that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space V such that $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$ (the ***cancellation law*** for vector addition). As illustrated, justify the steps by filling in the blanks.

$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$	Hypothesis
$(\mathbf{u}+\mathbf{w})+(-\mathbf{w})=(\mathbf{v}+\mathbf{w})+(-\mathbf{w})$	Add $-\mathbf{w}$ to both sides.
$\mathbf{u}+[\mathbf{w}+(-\mathbf{w})]=\mathbf{v}+[\mathbf{w}+(-\mathbf{w})]$	__________
$\mathbf{u}+\mathbf{0}=\mathbf{v}+\mathbf{0}$	__________
$\mathbf{u}=\mathbf{v}$	__________

22. Below is a seven-step proof of part (*b*) of Theorem 4.1.1. Justify each step either by stating that it is true by *hypothesis* or by specifying which of the ten vector space axioms applies.

Hypothesis: Let $\mathbf{u}$ be any vector in a vector space V, let $\mathbf{0}$ be the zero vector in V, and let k be a scalar.

Conclusion: Then $k\mathbf{0}=\mathbf{0}$.

Proof: (1) $k\mathbf{0}+k\mathbf{u}=k(\mathbf{0}+\mathbf{u})$
(2) $\qquad\qquad = k\mathbf{u}$
(3) Since $k\mathbf{u}$ is in V, $-k\mathbf{u}$ is in V.
(4) Therefore, $(k\mathbf{0}+k\mathbf{u})+(-k\mathbf{u})=k\mathbf{u}+(-k\mathbf{u})$.
(5) $k\mathbf{0}+(k\mathbf{u}+(-k\mathbf{u}))=k\mathbf{u}+(-k\mathbf{u})$
(6) $k\mathbf{0}+\mathbf{0}=\mathbf{0}$
(7) $k\mathbf{0}=\mathbf{0}$

In Exercises 23–24, let $\mathbf{u}$ be any vector in a vector space V. Give a step-by-step proof of the stated result using Exercises 21 and 22 as models for your presentation.

23. $0\mathbf{u}=\mathbf{0}$

24. $-\mathbf{u}=(-1)\mathbf{u}$

In Exercises 25–27, prove that the given set with the stated operations is a vector space.

25. The set $V=\{\mathbf{0}\}$ with the operations of addition and scalar multiplication given in Example 1.

26. The set R^∞ of all infinite sequences of real numbers with the operations of addition and scalar multiplication given in Example 3.

27. The set M_{mn} of all $m\times n$ matrices with the usual operations of addition and scalar multiplication.

28. Prove: If $\mathbf{u}$ is a vector in a vector space V and k a scalar such that $k\mathbf{u}=\mathbf{0}$, then either $k=0$ or $\mathbf{u}=\mathbf{0}$. [*Suggestion:* Show that if $k\mathbf{u}=\mathbf{0}$ and $k\neq 0$, then $\mathbf{u}=\mathbf{0}$. The result then follows as a logical consequence of this.]

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) A vector is any element of a vector space.

(b) A vector space must contain at least two vectors.

(c) If $\mathbf{u}$ is a vector and k is a scalar such that $k\mathbf{u}=\mathbf{0}$, then it must be true that $k=0$.

(d) The set of positive real numbers is a vector space if vector addition and scalar multiplication are the usual operations of addition and multiplication of real numbers.

(e) In every vector space the vectors $(-1)\mathbf{u}$ and $-\mathbf{u}$ are the same.

(f) In the vector space $F(-\infty,\infty)$ any function whose graph passes through the origin is a zero vector.

4.2 Subspaces

It is often the case that some vector space of interest is contained within a larger vector space whose properties are known. In this section we will show how to recognize when this is the case, we will explain how the properties of the larger vector space can be used to obtain properties of the smaller vector space, and we will give a variety of important examples.

We begin with some terminology.

> **DEFINITION 1** A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, to show that a nonempty set W with two operations is a vector space one must verify the ten vector space axioms. However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are "inherited" from V. For example, it is *not* necessary to verify that $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ holds in W because it holds for all vectors in V including those in W. On the other hand, it *is* necessary to verify

The Linear Transformation Viewpoint

Theorem 4.2.4 can be viewed as a statement about matrix transformations by letting $T_A: R^n \to R^m$ be multiplication by the coefficient matrix A. From this point of view the solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is sometimes called the ***kernel*** of the transformation, so with this terminology Theorem 4.2.4 can be rephrased as follows.

THEOREM 4.2.5 *If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A: R^n \to R^m$ is a subspace of R^n.*

A Concluding Observation

It is important to recognize that spanning sets are not unique. For example, any nonzero vector on the line in Figure 4.2.6*a* will span that line, and any two noncollinear vectors in the plane in Figure 4.2.6*b* will span that plane. The following theorem, whose proof is left as an exercise, states conditions under which two sets of vectors will span the same space.

THEOREM 4.2.6 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then*

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

Exercise Set 4.2

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3.
   (a) All vectors of the form $(a, 0, 0)$.
   (b) All vectors of the form $(a, 1, 1)$.
   (c) All vectors of the form (a, b, c), where $b = a + c$.
   (d) All vectors of the form (a, b, c), where $b = a + c + 1$.
   (e) All vectors of the form $(a, b, 0)$.

2. Use Theorem 4.2.1 to determine which of the following are subspaces of M_{nn}.
   (a) The set of all diagonal $n \times n$ matrices.
   (b) The set of all $n \times n$ matrices A such that $\det(A) = 0$.
   (c) The set of all $n \times n$ matrices A such that $\text{tr}(A) = 0$.
   (d) The set of all symmetric $n \times n$ matrices.
   (e) The set of all $n \times n$ matrices A such that $A^T = -A$.
   (f) The set of all $n \times n$ matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
   (g) The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed $n \times n$ matrix B.

3. Use Theorem 4.2.1 to determine which of the following are subspaces of P_3.
   (a) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
   (b) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
   (c) All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.
   (d) All polynomials of the form $a_0 + a_1x$, where a_0 and a_1 are real numbers.

4. Which of the following are subspaces of $F(-\infty, \infty)$?
   (a) All functions f in $F(-\infty, \infty)$ for which $f(0) = 0$.
   (b) All functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.
   (c) All functions f in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.
   (d) All polynomials of degree 2.

5. Which of the following are subspaces of R^∞?
   (a) All sequences $\mathbf{v}$ in R^∞ of the form $\mathbf{v} = (v, 0, v, 0, v, 0, \ldots)$.

(b) All sequences $\mathbf{v}$ in R^∞ of the form
$\mathbf{v} = (v, 1, v, 1, v, 1, \ldots)$.

(c) All sequences $\mathbf{v}$ in R^∞ of the form
$\mathbf{v} = (v, 2v, 4v, 8v, 16v, \ldots)$.

(d) All sequences in R^∞ whose components are 0 from some point on.

6. A line L through the origin in R^3 can be represented by parametric equations of the form $x = at$, $y = bt$, and $z = ct$. Use these equations to show that L is a subspace of R^3 by showing that if $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ are points on L and k is any real number, then $k\mathbf{v}_1$ and $\mathbf{v}_1 + \mathbf{v}_2$ are also points on L.

7. Which of the following are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

(a) $(2, 2, 2)$ (b) $(0, 4, 5)$ (c) $(0, 0, 0)$

8. Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

(a) $(-9, -7, -15)$ (b) $(6, 11, 6)$ (c) $(0, 0, 0)$

9. Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

10. In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

(a) $-9 - 7x - 15x^2$ (b) $6 + 11x + 6x^2$

(c) 0 (d) $7 + 8x + 9x^2$

11. In each part, determine whether the vectors span R^3.

(a) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1)$

(b) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

12. Suppose that $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, and $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(2, 3, -7, 3)$ (b) $(0, 0, 0, 0)$

(c) $(1, 1, 1, 1)$ (d) $(-4, 6, -13, 4)$

13. Determine whether the following polynomials span P_2.

$$\mathbf{p}_1 = 1 - x + 2x^2, \quad \mathbf{p}_2 = 3 + x,$$
$$\mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

14. Let $\mathbf{f} = \cos^2 x$ and $\mathbf{g} = \sin^2 x$. Which of the following lie in the space spanned by $\mathbf{f}$ and $\mathbf{g}$?

(a) $\cos 2x$ (b) $3 + x^2$ (c) 1 (d) $\sin x$ (e) 0

15. Determine whether the solution space of the system $A\mathbf{x} = \mathbf{0}$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

16. (*Calculus required*) Show that the following sets of functions are subspaces of $F(-\infty, \infty)$.

(a) All continuous functions on $(-\infty, \infty)$.

(b) All differentiable functions on $(-\infty, \infty)$.

(c) All differentiable functions on $(-\infty, \infty)$ that satisfy $\mathbf{f}' + 2\mathbf{f} = \mathbf{0}$.

17. (*Calculus required*) Show that the set of continuous functions $\mathbf{f} = f(x)$ on $[a, b]$ such that

$$\int_a^b f(x)\,dx = 0$$

is a subspace of $C[a, b]$.

18. Show that the solution vectors of a consistent nonhomogeneous system of m linear equations in n unknowns do not form a subspace of R^n.

19. In each part, let $T_A\colon R^2 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (1, 2)$ and $\mathbf{u}_2 = (-1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2)\}$ spans R^2.

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

20. In each part, let $T_A\colon R^3 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (0, 1, 1)$ and $\mathbf{u}_2 = (2, -1, 1)$ and $\mathbf{u}_3 = (1, 1, -2)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ spans R^2.

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}$

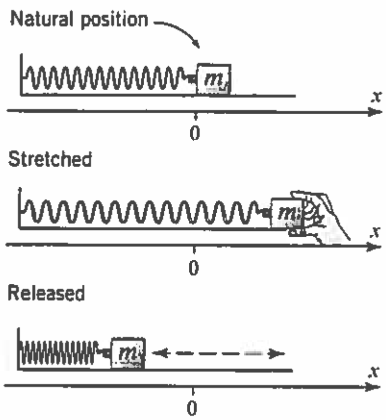
21. If T_A is multiplication by a matrix A with three columns, then the kernel of T_A is one of four possible geometric objects. What are they? Explain how you reached your conclusion.

22. Let $\mathbf{v}_1 = (1, 6, 4)$, $\mathbf{v}_2 = (2, 4, -1)$, $\mathbf{v}_3 = (-1, 2, 5)$, and $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 8, 9)$. Use Theorem 4.2.6 to show that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

23. The accompanying figure shows a mass-spring system in which a block of mass m is set into vibratory motion by pulling the block beyond its natural position at $x = 0$ and releasing it at time $t = 0$. If friction and air resistance are ignored, then the x-coordinate $x(t)$ of the block at time t is given by a function of the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

where ω is a fixed constant that depends on the mass of the block and the stiffness of the spring and c_1 and c_2 are arbitrary. Show that this set of functions forms a subspace of $C^\infty(-\infty, \infty)$.



▲ Figure Ex-23

Working with Proofs

24. Prove Theorem 4.2.6.

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Every subspace of a vector space is itself a vector space.

(b) Every vector space is a subspace of itself.

(c) Every subset of a vector space V that contains the zero vector in V is a subspace of V.

(d) The kernel of a matrix transformation $T_A: R^n \to R^m$ is a subspace of R^m.

(e) The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns is a subspace of R^n.

(f) The span of any finite set of vectors in a vector space is closed under addition and scalar multiplication.

(g) The intersection of any two subspaces of a vector space V is a subspace of V.

(h) The union of any two subspaces of a vector space V is a subspace of V.

(i) Two subsets of a vector space V that span the same subspace of V must be equal.

(j) The set of upper triangular $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrices.

(k) The polynomials $x - 1$, $(x-1)^2$, and $(x-1)^3$ span P_3.

Working with Technology

T1. Recall from Theorem 1.3.1 that a product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of the matrix A in which the coefficients are the entries of $\mathbf{x}$. Use matrix multiplication to compute

$$\mathbf{v} = 6(8, -2, 1, -4) + 17(-3, 9, 11, 6) - 9(13, -1, 2, 4)$$

T2. Use the idea in Exercise T1 and matrix multiplication to determine whether the polynomial

$$\mathbf{p} = 1 + x + x^2 + x^3$$

is in the span of

$$\mathbf{p}_1 = 8 - 2x + x^2 - 4x^3, \quad \mathbf{p}_2 = -3 + 9x + 11x^2 + 6x^3,$$
$$\mathbf{p}_3 = 13 - x + 2x^2 + 4x^3$$

T3. For the vectors that follow, determine whether

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

$$\mathbf{v}_1 = (-1, 2, 0, 1, 3), \quad \mathbf{v}_2 = (7, 4, 6, -3, 1),$$
$$\mathbf{v}_3 = (-5, 3, 1, 2, 4)$$
$$\mathbf{w}_1 = (-6, 5, 1, 3, 7), \quad \mathbf{w}_2 = (6, 6, 6, -2, 4),$$
$$\mathbf{w}_3 = (2, 7, 7, -1, 5)$$

4.3 Linear Independence

In this section we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of the others. This is important to know in applications because the existence of such relationships often signals that some kind of complication is likely to occur.

Linear Independence and Dependence

In a rectangular xy-coordinate system every vector in the plane can be expressed in exactly one way as a linear combination of the standard unit vectors. For example, the only way to express the vector $(3, 2)$ as a linear combination of $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ is

$$(3, 2) = 3(1, 0) + 2(0, 1) = 3\mathbf{i} + 2\mathbf{j} \tag{1}$$