▶ EXAMPLE 8 Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = e^x$, and $\mathbf{f}_3 = e^{2x}$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x}$$

This function is obviously not identically zero on $(-\infty, \infty)$, so $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ form a linearly independent set. ◀

OPTIONAL

We will close this section by proving Theorem 4.3.1.

Proof of Theorem 4.3.1 We will prove this theorem in the case where the set S has two or more vectors, and leave the case where S has only one vector as an exercise. Assume first that S is linearly independent. We will show that if the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0} \tag{11}$$

can be satisfied with coefficients that are not all zero, then at least one of the vectors in S must be expressible as a linear combination of the others, thereby contradicting the assumption of linear independence. To be specific, suppose that $k_1 \neq 0$. Then we can rewrite (11) as

$$\mathbf{v}_1 = \left(-\frac{k_2}{k_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{k_r}{k_1}\right)\mathbf{v}_r$$

which expresses $\mathbf{v}_1$ as a linear combination of the other vectors in S.

Conversely, we must show that if the only coefficients satisfying (11) are

$$k_1 = 0, \quad k_2 = 0, \ldots, \quad k_r = 0$$

then the vectors in S must be linearly independent. But if this were true of the coefficients and the vectors were not linearly independent, then at least one of them would be expressible as a linear combination of the others, say

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r$$

which we can rewrite as

$$\mathbf{v}_1 + (-c_2)\mathbf{v}_2 + \cdots + (-c_r)\mathbf{v}_r = \mathbf{0}$$

But this contradicts our assumption that (11) can only be satisfied by coefficients that are all zero. Thus, the vectors in S must be linearly independent. ◀

Exercise Set 4.3

1. Explain why the following form linearly dependent sets of vectors. (Solve this problem by inspection.)

   (a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3

   (b) $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2

   (c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2

   (d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

2. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^3.

   (a) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

   (b) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

3. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

   (a) $(3, 8, 7, -3)$, $(1, 5, 3, -1)$, $(2, -1, 2, 6)$, $(4, 2, 6, 4)$

   (b) $(3, 0, -3, 6)$, $(0, 2, 3, 1)$, $(0, -2, -2, 0)$, $(-2, 1, 2, 1)$

4. In each part, determine whether the vectors are linearly independent or are linearly dependent in P_2.

(a) $2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$

(b) $1 + 3x + 3x^2,\ x + 4x^2,\ 5 + 6x + 3x^2,\ 7 + 2x - x^2$

5. In each part, determine whether the matrices are linearly independent or dependent.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}

6. Determine all values of k for which the following matrices are linearly independent in M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix},\ \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix},\ \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

7. In each part, determine whether the three vectors lie in a plane in R^3.

(a) $\mathbf{v}_1 = (2, -2, 0),\ \mathbf{v}_2 = (6, 1, 4),\ \mathbf{v}_3 = (2, 0, -4)$

(b) $\mathbf{v}_1 = (-6, 7, 2),\ \mathbf{v}_2 = (3, 2, 4),\ \mathbf{v}_3 = (4, -1, 2)$

8. In each part, determine whether the three vectors lie on the same line in R^3.

(a) $\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (2, -4, -6),\ \mathbf{v}_3 = (-3, 6, 0)$

(b) $\mathbf{v}_1 = (2, -1, 4),\ \mathbf{v}_2 = (4, 2, 3),\ \mathbf{v}_3 = (2, 7, -6)$

(c) $\mathbf{v}_1 = (4, 6, 8),\ \mathbf{v}_2 = (2, 3, 4),\ \mathbf{v}_3 = (-2, -3, -4)$

9. (a) Show that the three vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$, and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

10. (a) Show that the vectors $\mathbf{v}_1 = (1, 2, 3, 4)$, $\mathbf{v}_2 = (0, 1, 0, -1)$, and $\mathbf{v}_3 = (1, 3, 3, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

11. For which real values of λ do the following vectors form a linearly dependent set in R^3?

$$\mathbf{v}_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right),\quad \mathbf{v}_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right),\quad \mathbf{v}_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

12. Under what conditions is a set with one vector linearly independent?

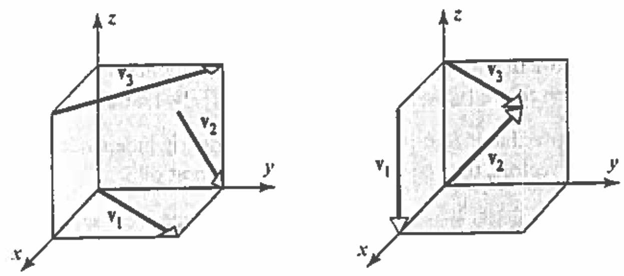
13. In each part, let $T_A\colon R^2 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (1, 2)$ and $\mathbf{u}_2 = (-1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2)\}$ is linearly independent in R^2.

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

14. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A, and let $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (2, -1, 1)$, and $\mathbf{u}_3 = (0, 1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent in R^3.

(a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

15. Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in part (*a*) of the accompanying figure linearly independent? What about those in part (*b*)? Explain.



(*a*) (*b*)

▲ Figure Ex-15

16. By using appropriate identities, where required, determine which of the following sets of vectors in $F(-\infty, \infty)$ are linearly dependent.

(a) $6,\ 3\sin^2 x,\ 2\cos^2 x$ (b) $x,\ \cos x$

(c) $1,\ \sin x,\ \sin 2x$ (d) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

(e) $(3 - x)^2,\ x^2 - 6x,\ 5$ (f) $0,\ \cos^3 \pi x,\ \sin^5 3\pi x$

17. ***(Calculus required)*** The functions

$$f_1(x) = x \quad \text{and} \quad f_2(x) = \cos x$$

are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using the Wronskian.

18. ***(Calculus required)*** The functions

$$f_1(x) = \sin x \quad \text{and} \quad f_2(x) = \cos x$$

are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using the Wronskian.

19. ***(Calculus required)*** Use the Wronskian to show that the following sets of vectors are linearly independent.

(a) $1,\ x,\ e^x$ (b) $1,\ x,\ x^2$

20. ***(Calculus required)*** Use the Wronskian to show that the functions $f_1(x) = e^x$, $f_2(x) = xe^x$, and $f_3(x) = x^2e^x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

21. ***(Calculus required)*** Use the Wronskian to show that the functions $f_1(x) = \sin x$, $f_2(x) = \cos x$, and $f_3(x) = x\cos x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

22. Show that for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in a vector space V, the vectors $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$, and $\mathbf{w}-\mathbf{u}$ form a linearly dependent set.

23. (a) In Example 1 we showed that the mutually orthogonal vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a linearly independent set of vectors in R^3. Do you think that every set of three nonzero mutually orthogonal vectors in R^3 is linearly independent? Justify your conclusion with a geometric argument.

(b) Justify your conclusion with an algebraic argument. [*Hint:* Use dot products.]

Working with Proofs

24. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then so are $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1\}$, $\{\mathbf{v}_2\}$, and $\{\mathbf{v}_3\}$.

25. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

26. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V, and $\mathbf{v}_4$ is any vector in V that is not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

27. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set of vectors in a vector space V, and if $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ are any vectors in V that are not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is also linearly dependent.

28. Prove that in P_2 every set with more than three vectors is linearly dependent.

29. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

30. Use part (a) of Theorem 4.3.1 to prove part (b).

31. Prove part (b) of Theorem 4.3.2.

32. Prove part (c) of Theorem 4.3.2.

True-False Exercises

TF. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A set containing a single vector is linearly independent.

(b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k.

(c) Every linearly dependent set contains the zero vector.

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar k.

(e) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.

(f) The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}.

(g) The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent.

(h) The functions f_1 and f_2 are linearly dependent if there is a real number x such that $k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2.

Working with Technology

T1. Devise three different methods for using your technology utility to determine whether a set of vectors in R^n is linearly independent, and then use each of those methods to determine whether the following vectors are linearly independent.

$$\mathbf{v}_1 = (4, -5, 2, 6), \quad \mathbf{v}_2 = (2, -2, 1, 3),$$
$$\mathbf{v}_3 = (6, -3, 3, 9), \quad \mathbf{v}_4 = (4, -1, 5, 6)$$

T2. Show that $S = \{\cos t, \sin t, \cos 2t, \sin 2t\}$ is a linearly independent set in $C(-\infty, \infty)$ by evaluating the left side of the equation

$$c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t = 0$$

at sufficiently many values of t to obtain a linear system whose only solution is $c_1 = c_2 = c_3 = c_4 = 0$.

4.4 Coordinates and Basis

We usually think of a line as being one-dimensional, a plane as two-dimensional, and the space around us as three-dimensional. It is the primary goal of this section and the next to make this intuitive notion of dimension precise. In this section we will discuss coordinate systems in general vector spaces and lay the groundwork for a precise definition of dimension in the next section.

Coordinate Systems in Linear Algebra

In analytic geometry one uses *rectangular* coordinate systems to create a one-to-one correspondence between points in 2-space and ordered pairs of real numbers and between points in 3-space and ordered triples of real numbers (Figure 4.4.1). Although rectangular coordinate systems are common, they are not essential. For example, Figure 4.4.2 shows coordinate systems in 2-space and 3-space in which the coordinate axes are not mutually perpendicular.

or, in terms of components,

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

Equating corresponding components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 \qquad\quad + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain $c_1 = 1$, $c_2 = -1$, $c_3 = 2$ (verify). Therefore,

$$(\mathbf{v})_S = (1, -1, 2)$$

Solution (b) Using the definition of $(\mathbf{v})_S$, we obtain

$$\begin{aligned} \mathbf{v} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) = (11, 31, 7) \end{aligned}$$ ◀

Exercise Set 4.4

1. Use the method of Example 3 to show that the following set of vectors forms a basis for R^2.

$$\{(2, 1), (3, 0)\}$$

2. Use the method of Example 3 to show that the following set of vectors forms a basis for R^3.

$$\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

3. Show that the following polynomials form a basis for P_2.

$$x^2 + 1, \quad x^2 - 1, \quad 2x - 1$$

4. Show that the following polynomials form a basis for P_3.

$$1 + x, \quad 1 - x, \quad 1 - x^2, \quad 1 - x^3$$

5. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

6. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

7. In each part, show that the set of vectors is not a basis for R^3.
   (a) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$
   (b) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

8. Show that the following vectors do not form a basis for P_2.

$$1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$$

9. Show that the following matrices do not form a basis for M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

10. Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$, $\mathbf{v}_3 = \cos 2x$.
    (a) Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for V.
    (b) Find a basis for V.

11. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.
    (a) $\mathbf{u}_1 = (2, -4)$, $\mathbf{u}_2 = (3, 8)$; $\mathbf{w} = (1, 1)$
    (b) $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (0, 2)$; $\mathbf{w} = (a, b)$

12. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.
    (a) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 0)$
    (b) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (0, 1)$

13. Find the coordinate vector of $\mathbf{v}$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3.
    (a) $\mathbf{v} = (2, -1, 3)$; $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$, $\mathbf{v}_3 = (3, 3, 3)$
    (b) $\mathbf{v} = (5, -12, 3)$; $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

14. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2.
    (a) $\mathbf{p} = 4 - 3x + x^2$; $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$
    (b) $\mathbf{p} = 2 - x + x^2$; $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$

In Exercises 15–16, first show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

15. $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$,
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

16. $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$

In Exercises 17–18, first show that the set $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for P_2, then express $\mathbf{p}$ as a linear combination of the vectors in S, and then find the coordinate vector of $\mathbf{p}$ relative to S.

17. $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x + x^2$, $\mathbf{p}_3 = x^2$;
$\mathbf{p} = 7 - x + 2x^2$

18. $\mathbf{p}_1 = 1 + 2x + x^2$, $\mathbf{p}_2 = 2 + 9x$, $\mathbf{p}_3 = 3 + 3x + 4x^2$;
$\mathbf{p} = 2 + 17x - 3x^2$

19. In words, explain why the sets of vectors in parts (a) to (d) are *not* bases for the indicated vector spaces.

(a) $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (0, 3)$, $\mathbf{u}_3 = (1, 5)$ for R^2

(b) $\mathbf{u}_1 = (-1, 3, 2)$, $\mathbf{u}_2 = (6, 1, 1)$ for R^3

(c) $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x$ for P_2

(d) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}$,
$D = \begin{bmatrix} 5 & 0 \\ 4 & 2 \end{bmatrix}$ for M_{22}

20. In any vector space a set that contains the zero vector must be linearly dependent. Explain why this is so.

21. In each part, let $T_A: R^3 \to R^3$ be multiplication by A, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for R^3. Determine whether the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly independent in R^2.

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

22. In each part, let $T_A: R^3 \to R^3$ be multiplication by A, and let $\mathbf{u} = (1, -2, -1)$. Find the coordinate vector of $T_A(\mathbf{u})$ relative to the basis $S = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ for R^3.

(a) $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

23. The accompanying figure shows a rectangular xy-coordinate system determined by the unit basis vectors $\mathbf{i}$ and $\mathbf{j}$ and an $x'y'$-coordinate system determined by unit basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Find the $x'y'$-coordinates of the points whose xy-coordinates are given.

(a) $(\sqrt{3}, 1)$ (b) $(1, 0)$ (c) $(0, 1)$ (d) (a, b)

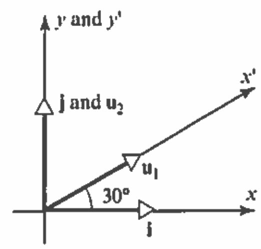


Figure Ex-23

24. The accompanying figure shows a rectangular xy-coordinate system and an $x'y'$-coordinate system with skewed axes. Assuming that 1-unit scales are used on all the axes, find the $x'y'$-coordinates of the points whose xy-coordinates are given.

(a) $(1, 1)$ (b) $(1, 0)$ (c) $(0, 1)$ (d) (a, b)

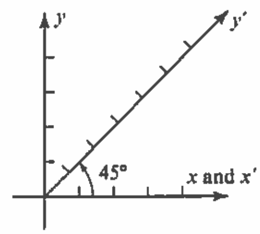


Figure Ex-24

25. The first four ***Hermite polynomials*** [named for the French mathematician Charles Hermite (1822–1901)] are

$$1, \quad 2t, \quad -2 + 4t^2, \quad -12t + 8t^3$$

These polynomials have a wide variety of applications in physics and engineering.

(a) Show that the first four Hermite polynomials form a basis for P_3.

(b) Let B be the basis in part (a). Find the coordinate vector of the polynomial

$$\mathbf{p}(t) = -1 - 4t + 8t^2 + 8t^3$$

relative to B.

26. The first four ***Laguerre polynomials*** [named for the French mathematician Edmond Laguerre (1834–1886)] are

$$1, \quad 1 - t, \quad 2 - 4t + t^2, \quad 6 - 18t + 9t^2 - t^3$$

(a) Show that the first four Laguerre polynomials form a basis for P_3.

(b) Let B be the basis in part (a). Find the coordinate vector of the polynomial

$$\mathbf{p}(t) = -10t + 9t^2 - t^3$$

relative to B.

27. Consider the coordinate vectors

$$[\mathbf{w}]_S = \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}, \quad [\mathbf{q}]_S = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad [B]_S = \begin{bmatrix} -8 \\ 7 \\ 6 \\ 3 \end{bmatrix}$$

(a) Find $\mathbf{w}$ if S is the basis in Exercise 2.

(b) Find $\mathbf{q}$ if S is the basis in Exercise 3.

(c) Find B if S is the basis in Exercise 5.

28. The basis that we gave for M_{22} in Example 4 consisted of non-invertible matrices. Do you think that there is a basis for M_{22} consisting of invertible matrices? Justify your answer.

Working with Proofs

29. Prove that R^∞ is an infinite-dimensional vector space.

30. Let $T_A: R^n \to R^n$ be multiplication by an invertible matrix A, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for R^n. Prove that $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), \ldots, T_A(\mathbf{u}_n)\}$ is also a basis for R^n.

31. Prove that if V is a subspace of a vector space W and if V is infinite-dimensional, then so is W.

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V.

(b) Every linearly independent subset of a vector space V is a basis for V.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

(d) The coordinate vector of a vector $\mathbf{x}$ in R^n relative to the standard basis for R^n is $\mathbf{x}$.

(e) Every basis of P_4 contains at least one polynomial of degree 3 or less.

Working with Technology

T1. Let V be the subspace of P_3 spanned by the vectors

$$\mathbf{p}_1 = 1 + 5x - 3x^2 - 11x^3, \quad \mathbf{p}_2 = 7 + 4x - x^2 + 2x^3,$$
$$\mathbf{p}_3 = 5 + x + 9x^2 + 2x^3, \quad \mathbf{p}_4 = 3 - x + 7x^2 + 5x^3$$

(a) Find a basis S for V.

(b) Find the coordinate vector of $\mathbf{p} = 19 + 18x - 13x^2 - 10x^3$ relative to the basis S you obtained in part (a).

T2. Let V be the subspace of $C^\infty(-\infty, \infty)$ spanned by the vectors in the set

$$B = \{1, \cos x, \cos^2 x, \cos^3 x, \cos^4 x, \cos^5 x\}$$

and accept without proof that B is a basis for V. Confirm that the following vectors are in V, and find their coordinate vectors relative to B.

$$\mathbf{f}_0 = 1, \quad \mathbf{f}_1 = \cos x, \quad \mathbf{f}_2 = \cos 2x, \quad \mathbf{f}_3 = \cos 3x,$$
$$\mathbf{f}_4 = \cos 4x, \quad \mathbf{f}_5 = \cos 5x$$

4.5 Dimension

We showed in the previous section that the standard basis for R^n has n vectors and hence that the standard basis for R^3 has three vectors, the standard basis for R^2 has two vectors, and the standard basis for $R^1 (= R)$ has one vector. Since we think of space as three-dimensional, a plane as two-dimensional, and a line as one-dimensional, there seems to be a link between the number of vectors in a basis and the dimension of a vector space. We will develop this idea in this section.

Number of Vectors in a Basis

Our first goal in this section is to establish the following fundamental theorem.

THEOREM 4.5.1 *All bases for a finite-dimensional vector space have the same number of vectors.*

To prove this theorem we will need the following preliminary result, whose proof is deferred to the end of the section.

Exercise Set 4.5

In Exercises 1–6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

1. $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \quad\quad + x_3 &= 0 \end{aligned}$$

2. $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

3. $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 \quad\quad + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

5. $$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

7. In each part, find a basis for the given subspace of R^3, and state its dimension.
   (a) The plane $3x - 2y + 5z = 0$.
   (b) The plane $x - y = 0$.
   (c) The line $x = 2t, y = -t, z = 4t$.
   (d) All vectors of the form (a, b, c), where $b = a + c$.

8. In each part, find a basis for the given subspace of R^4, and state its dimension.
   (a) All vectors of the form $(a, b, c, 0)$.
   (b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.
   (c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

9. Find the dimension of each of the following vector spaces.
   (a) The vector space of all diagonal $n \times n$ matrices.
   (b) The vector space of all symmetric $n \times n$ matrices.
   (c) The vector space of all upper triangular $n \times n$ matrices.

10. Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

11. (a) Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.
    (b) Make a conjecture about the dimension of W.
    (c) Confirm your conjecture by finding a basis for W.

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.
    (a) $\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (1, -2, -2)$
    (b) $\mathbf{v}_1 = (1, -1, 0),\ \mathbf{v}_2 = (3, 1, -2)$

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.
    $$\mathbf{v}_1 = (1, -4, 2, -3),\quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

15. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^3.

16. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4.

17. Find a basis for the subspace of R^3 that is spanned by the vectors
    $$\mathbf{v}_1 = (1, 0, 0),\quad \mathbf{v}_2 = (1, 0, 1),\quad \mathbf{v}_3 = (2, 0, 1),\quad \mathbf{v}_4 = (0, 0, -1)$$

18. Find a basis for the subspace of R^4 that is spanned by the vectors
    $$\mathbf{v}_1 = (1, 1, 1, 1),\quad \mathbf{v}_2 = (2, 2, 2, 0),\quad \mathbf{v}_3 = (0, 0, 0, 3),\quad \mathbf{v}_4 = (3, 3, 3, 4)$$

19. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A and find the dimension of the subspace of R^3 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.
    (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$
    (c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

20. In each part, let T_A be multiplication by A and find the dimension of the subspace R^4 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.
    (a) $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Working with Proofs

21. (a) Prove that for every positive integer n, one can find $n + 1$ linearly independent vectors in $F(-\infty, \infty)$. [*Hint:* Look for polynomials.]
    (b) Use the result in part (a) to prove that $F(-\infty, \infty)$ is infinite-dimensional.
    (c) Prove that $C(-\infty, \infty)$, $C^m(-\infty, \infty)$, and $C^\infty(-\infty, \infty)$ are infinite-dimensional.

22. Let S be a basis for an n-dimensional vector space V. Prove that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ form a linearly independent set in R^n, and conversely.

23. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a nonempty set of vectors in an n-dimensional vector space V. Prove that if the vectors in S span V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ span R^n, and conversely.

24. Prove part (*a*) of Theorem 4.5.6.

25. Prove: A subspace of a finite-dimensional vector space is finite-dimensional.

26. State the two parts of Theorem 4.5.2 in contrapositive form.

27. In each part, let S be the standard basis for P_2. Use the results proved in Exercises 22 and 23 to find a basis for the subspace of P_2 spanned by the given vectors.

    (a) $-1 + x - 2x^2,\ 3 + 3x + 6x^2,\ 9$

    (b) $1 + x,\ x^2,\ 2 + 2x + 3x^2$

    (c) $1 + x - 3x^2,\ 2 + 2x - 6x^2,\ 3 + 3x - 9x^2$

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) The zero vector space has dimension zero.

(b) There is a set of 17 linearly independent vectors in R^{17}.

(c) There is a set of 11 vectors that span R^{17}.

(d) Every linearly independent set of five vectors in R^5 is a basis for R^5.

(e) Every set of five vectors that spans R^5 is a basis for R^5.

(f) Every set of vectors that spans R^n contains a basis for R^n.

(g) Every linearly independent set of vectors in R^n is contained in some basis for R^n.

(h) There is a basis for M_{22} consisting of invertible matrices.

(i) If A has size $n \times n$ and $I_n, A, A^2, \ldots, A^{n^2}$ are distinct matrices, then $\{I_n, A, A^2, \ldots, A^{n^2}\}$ is a linearly dependent set.

(j) There are at least two distinct three-dimensional subspaces of P_2.

(k) There are only three distinct two-dimensional subspaces of P_2.

Working with Technology

T1. Devise three different procedures for using your technology utility to determine the dimension of the subspace spanned by a set of vectors in R^n, and then use each of those procedures to determine the dimension of the subspace of R^5 spanned by the vectors

$$\mathbf{v}_1 = (2, 2, -1, 0, 1), \quad \mathbf{v}_2 = (-1, -1, 2, -3, 1),$$
$$\mathbf{v}_3 = (1, 1, -2, 0, -1), \quad \mathbf{v}_4 = (0, 0, 1, 1, 1)$$

T2. Find a basis for the row space of A by starting at the top and successively removing each row that is a linear combination of its predecessors.

$$A = \begin{bmatrix} 3.4 & 2.2 & 1.0 & -1.8 \\ 2.1 & 3.6 & 4.0 & -3.4 \\ 8.9 & 8.0 & 6.0 & 7.0 \\ 7.6 & 9.4 & 9.0 & -8.6 \\ 1.0 & 2.2 & 0.0 & 2.2 \end{bmatrix}$$

4.6 Change of Basis

A basis that is suitable for one problem may not be suitable for another, so it is a common process in the study of vector spaces to change from one basis to another. Because a basis is the vector space generalization of a coordinate system, changing bases is akin to changing coordinate axes in R^2 and R^3. In this section we will study problems related to changing bases.

Coordinate Maps

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

is the coordinate vector of $\mathbf{v}$ relative to S, then, as illustrated in Figure 4.4.6, the mapping

$$\mathbf{v} \rightarrow (\mathbf{v})_S \tag{1}$$

creates a connection (a one-to-one correspondence) between vectors in the *general* vector space V and vectors in the *Euclidean* vector space R^n. We call (1) the ***coordinate map relative to S*** from V to R^n. In this section we will find it convenient to express coordinate

We call these the ***dependency equations***. The corresponding relationships in (5) are

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$
$$\mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4 \blacktriangleleft$$

The following is a summary of the steps that we followed in our last example to solve the problem posed above.

Basis for the Space Spanned by a Set of Vectors

Step 1. Form the matrix A whose columns are the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Step 2. Reduce the matrix A to reduced row echelon form R.

Step 3. Denote the column vectors of R by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.

Step 4. Identify the columns of R that contain the leading 1's. The corresponding column vectors of A form a basis for span(S).

This completes the first part of the problem.

Step 5. Obtain a set of dependency equations for the column vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ of R by successively expressing each $\mathbf{w}_i$ that does not contain a leading 1 of R as a linear combination of predecessors that do.

Step 6. In each dependency equation obtained in Step 5, replace the vector $\mathbf{w}_i$ by the vector $\mathbf{v}_i$ for $i = 1, 2, \ldots, k$.

This completes the second part of the problem.

Exercise Set 4.7

In Exercises 1–2, express the product $A\mathbf{x}$ as a linear combination of the column vectors of A.

1. (a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

2. (a) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$

In Exercises 3–4, determine whether $\mathbf{b}$ is in the column space of A, and if so, express $\mathbf{b}$ as a linear combination of the column vectors of A

3. (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

4. (a) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

5. Suppose that $x_1 = 3, x_2 = 0, x_3 = -1, x_4 = 5$ is a solution of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by the formulas

$$x_1 = 5r - 2s, \quad x_2 = s, \quad x_3 = s + t, \quad x_4 = t$$

   (a) Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

   (b) Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

6. Suppose that $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$ is a solution of a nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

   (a) Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

   (b) Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

In Exercises 7–8, find the vector form of the general solution of the linear system $A\mathbf{x} = \mathbf{b}$, and then use that result to find the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

7. (a) $\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_1 - 6x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{aligned}$

8. (a) $\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 &= -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 &= 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 &= -3 \end{aligned}$

(b) $\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 4 \\ -2x_1 + x_2 + 2x_3 + x_4 &= -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 3 \\ 4x_1 - 7x_2 - 5x_4 &= -5 \end{aligned}$

In Exercises 9–10, find bases for the null space and row space of A.

9. (a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

10. (a) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

In Exercises 11–12, a matrix in row echelon form is given. By inspection, find a basis for the row space and for the column space of that matrix.

11. (a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

12. (a) $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. (a) Use the methods of Examples 6 and 7 to find bases for the row space and column space of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

(b) Use the method of Example 9 to find a basis for the row space of A that consists entirely of row vectors of A.

In Exercises 14–15, find a basis for the subspace of R^4 that is spanned by the given vectors.

14. $(1, 1, -4, -3)$, $(2, 0, 2, -2)$, $(2, -1, 3, 2)$

15. $(1, 1, 0, 0)$, $(0, 0, 1, 1)$, $(-2, 0, 2, 2)$, $(0, -3, 0, 3)$

In Exericses 16–17, find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.

16. $\mathbf{v}_1 = (1, 0, 1, 1)$, $\mathbf{v}_2 = (-3, 3, 7, 1)$, $\mathbf{v}_3 = (-1, 3, 9, 3)$, $\mathbf{v}_4 = (-5, 3, 5, -1)$

17. $\mathbf{v}_1 = (1, -1, 5, 2)$, $\mathbf{v}_2 = (-2, 3, 1, 0)$, $\mathbf{v}_3 = (4, -5, 9, 4)$, $\mathbf{v}_4 = (0, 4, 2, -3)$, $\mathbf{v}_5 = (-7, 18, 2, -8)$

In Exercises 18–19, find a basis for the row space of A that consists entirely of row vectors of A.

18. The matrix in Exercise 10(a).

19. The matrix in Exercise 10(b).

20. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix}$$

21. In each part, let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 4 \end{bmatrix}$. For the given vector $\mathbf{b}$, find the general form of all vectors $\mathbf{x}$ in R^3 for which $T_A(\mathbf{x}) = \mathbf{b}$ if such vectors exist.

(a) $\mathbf{b} = (0, 0)$ (b) $\mathbf{b} = (1, 3)$ (c) $\mathbf{b} = (-1, 1)$

22. In each part, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$. For the given vector $\mathbf{b}$, find the general form of all vectors $\mathbf{x}$ in R^2 for which $T_A(\mathbf{x}) = \mathbf{b}$ if such vectors exist.

(a) $\mathbf{b} = (0, 0, 0, 0)$ (b) $\mathbf{b} = (1, 1, -1, -1)$

(c) $\mathbf{b} = (2, 0, 0, 2)$

23. (a) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that relative to an xyz-coordinate system in 3-space the null space of A consists of all points on the z-axis and that the column space consists of all points in the xy-plane (see the accompanying figure).

(b) Find a 3×3 matrix whose null space is the x-axis and whose column space is the yz-plane.

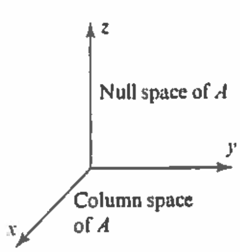


Figure Ex-23

24. Find a 3×3 matrix whose null space is

 (a) a point. (b) a line. (c) a plane.

25. (a) Find all 2×2 matrices whose null space is the line $3x - 5y = 0$.

 (b) Describe the null spaces of the following matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Working with Proofs

26. Prove Theorem 4.7.4.

27. Prove that the row vectors of an $n \times n$ invertible matrix A form a basis for R^n.

28. Suppose that A and B are $n \times n$ matrices and A is invertible. Invent and prove a theorem that describes how the row spaces of AB and B are related.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the column space of the matrix whose column vectors are $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(b) The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

(c) If R is the reduced row echelon form of A, then those column vectors of R that contain the leading 1's form a basis for the column space of A.

(d) The set of nonzero row vectors of a matrix A is a basis for the row space of A.

(e) If A and B are $n \times n$ matrices that have the same row space, then A and B have the same column space.

(f) If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the null space of EA is the same as the null space of A.

(g) If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the row space of EA is the same as the row space of A.

(h) If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the column space of EA is the same as the column space of A.

(i) The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is not in the column space of A.

(j) There is an invertible matrix A and a singular matrix B such that the row spaces of A and B are the same.

Working with Technology

T1. Find a basis for the column space of

$$A = \begin{bmatrix} 2 & 6 & 0 & 8 & 4 & 12 & 4 \\ 3 & 9 & -2 & 8 & 6 & 18 & 6 \\ 3 & 9 & -7 & -2 & 6 & -3 & -1 \\ 2 & 6 & 5 & 18 & 4 & 33 & 11 \\ 1 & 3 & -2 & 0 & 2 & 6 & 2 \end{bmatrix}$$

that consists of column vectors of A.

T2. Find a basis for the row space of the matrix A in Exercise T1 that consists of row vectors of A.

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

In the last section we investigated relationships between a system of linear equations and the row space, column space, and null space of its coefficient matrix. In this section we will be concerned with the dimensions of those spaces. The results we obtain will provide a deeper insight into the relationship between a linear system and its coefficient matrix.

Row and Column Spaces Have Equal Dimensions

In Examples 6 and 7 of Section 4.7 we found that the row and column spaces of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

both have three basis vectors and hence are both three-dimensional. The fact that these spaces have the same dimension is not accidental, but rather a consequence of the following theorem.

system by Gauss–Jordan elimination. We leave it for you to show that the augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 3b_2 + 2b_1 \\ 0 & 0 & b_4 - 4b_2 + 3b_1 \\ 0 & 0 & b_5 - 5b_2 + 4b_1 \end{bmatrix} \tag{7}$$

Thus, the system is consistent if and only if b_1, b_2, b_3, b_4, and b_5 satisfy the conditions

$$\begin{aligned} 2b_1 - 3b_2 + b_3 \qquad\qquad &= 0 \\ 3b_1 - 4b_2 \qquad + b_4 \qquad &= 0 \\ 4b_1 - 5b_2 \qquad\qquad + b_5 &= 0 \end{aligned}$$

Solving this homogeneous linear system yields

$$b_1 = 5r - 4s, \quad b_2 = 4r - 3s, \quad b_3 = 2r - s, \quad b_4 = r, \quad b_5 = s$$

where r and s are arbitrary. ◀

Remark The coefficient matrix for the given linear system in the last example has $n = 2$ columns, and it has rank $r = 2$ because there are two nonzero rows in its reduced row echelon form. This implies that when the system is consistent its general solution will contain $n - r = 0$ parameters; that is, the solution will be unique. With a moment's thought, you should be able to see that this is so from (7).

Exercise Set 4.8

► In Exercises 1–2, find the rank and nullity of the matrix A by reducing it to row echelon form. ◄

1. (a) $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

2. (a) $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$

► In Exercises 3–6, the matrix R is the reduced row echelon form of the matrix A.

(a) By inspection of the matrix R, find the rank and nullity of A.

(b) Confirm that the rank and nullity satisfy Formula (4).

(c) Find the number of leading variables and the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ without solving the system. ◄

3. $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & -6 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{bmatrix}$; $R = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. In each part, find the largest possible value for the rank of A and the smallest possible value for the nullity of A.

(a) A is 4×4 (b) A is 3×5 (c) A is 5×3

8. If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value for its nullity?

9. In each part, use the information in the table to:

(i) find the dimensions of the row space of A, column space of A, null space of A, and null space of A^T;

(ii) determine whether or not the linear system $A\mathbf{x} = \mathbf{b}$ is consistent;

(iii) find the number of parameters in the general solution of each system in (ii) that is consistent.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank[A | $\mathbf{b}$]	3	3	1	2	3	0	2

10. Verify that $\text{rank}(A) = \text{rank}(A^T)$.

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

11. (a) Find an equation relating nullity(A) and nullity(A^T) for the matrix in Exercise 10.

(b) Find an equation relating nullity(A) and nullity(A^T) for a general $m \times n$ matrix.

12. Let $T: R^2 \to R^3$ be the linear transformation defined by the formula

$$T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$$

(a) Find the rank of the standard matrix for T.

(b) Find the nullity of the standard matrix for T.

13. Let $T: R^5 \to R^3$ be the linear transformation defined by the formula

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_2 + x_3 + x_4, x_4 + x_5)$$

(a) Find the rank of the standard matrix for T.

(b) Find the nullity of the standard matrix for T.

14. Discuss how the rank of A varies with t.

(a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

15. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

16. (a) Give an example of a 3×3 matrix whose column space is a plane through the origin in 3-space.

(b) What kind of geometric object is the null space of your matrix?

(c) What kind of geometric object is the row space of your matrix?

17. Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

18. (a) If A is a 3×5 matrix, then the rank of A is at most ______. Why?

(b) If A is a 3×5 matrix, then the nullity of A is at most ______. Why?

(c) If A is a 3×5 matrix, then the rank of A^T is at most ______. Why?

(d) If A is a 3×5 matrix, then the nullity of A^T is at most ______. Why?

19. (a) If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ______. Why?

(b) If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ______. Why?

(c) If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ______. Why?

(d) If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ______. Why?

20. Let A be a 7×6 matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Find the rank and nullity of A.

21. Let A be a 5×7 matrix with rank 4.

(a) What is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

(b) Is $A\mathbf{x} = \mathbf{b}$ consistent for all vectors $\mathbf{b}$ in R^5? Explain.

22. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Show that A has rank 2 if and only if one or more of the following determinants is nonzero.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

23. Use the result in Exercise 22 to show that the set of points (x, y, z) in R^3 for which the matrix

$$\begin{bmatrix} x & y & z \\ 1 & x & y \end{bmatrix}$$

has rank 1 is the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$.

24. Find matrices A and B for which $\text{rank}(A) = \text{rank}(B)$, but $\text{rank}(A^2) \neq \text{rank}(B^2)$.

25. In Example 6 of Section 3.4 we showed that the row space and the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

are orthogonal complements in R^6, as guaranteed by part (*a*) of Theorem 4.8.7. Show that null space of A^T and the column space of A are orthogonal complements in R^4, as guaranteed by part (*b*) of Theorem 4.8.7. [*Suggestion:* Show that each column vector of A is orthogonal to each vector in a basis for the null space of A^T.]

26. Confirm the results stated in Theorem 4.8.7 for the matrix.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

27. In each part, state whether the system is overdetermined or underdetermined. If overdetermined, find all values of the b's for which it is inconsistent, and if underdetermined, find all values of the b's for which it is inconsistent and all values for which it has infinitely many solutions.

(a) $\begin{bmatrix} 1 & -1 \\ -3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 & 4 \\ -2 & -6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

28. What conditions must be satisfied by b_1, b_2, b_3, b_4, and b_5 for the overdetermined linear system

$$\begin{aligned} x_1 - 3x_2 &= b_1 \\ x_1 - 2x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 - 4x_2 &= b_4 \\ x_1 + 5x_2 &= b_5 \end{aligned}$$

to be consistent?

Working with Proofs

29. Prove: If $k \neq 0$, then A and kA have the same rank.

30. Prove: If a matrix A is not square, then either the row vectors or the column vectors of A are linearly dependent.

31. Use Theorem 4.8.3 to prove Theorem 4.8.4.

32. Prove Theorem 4.8.7(*b*).

33. Prove: If a vector $\mathbf{v}$ in R^n is orthogonal to each vector in a basis for a subspace W of R^n, then $\mathbf{v}$ is orthogonal to every vector in W.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) Either the row vectors or the column vectors of a square matrix are linearly independent.

(b) A matrix with linearly independent row vectors and linearly independent column vectors is square.

(c) The nullity of a nonzero $m \times n$ matrix is at most m.

(d) Adding one additional column to a matrix increases its rank by one.

(e) The nullity of a square matrix with linearly dependent rows is at least one.

(f) If A is square and $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then the nullity of A is zero.

(g) If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space.

(h) If $\text{rank}(A^T) = \text{rank}(A)$, then A is square.

(i) There is no 3×3 matrix whose row space and null space are both lines in 3-space.

(j) If V is a subspace of R^n and W is a subspace of V, then $W^\perp$ is a subspace of $V^\perp$.

Working with Technology

T1. It can be proved that a nonzero matrix A has rank k if and only if some $k \times k$ submatrix has a nonzero determinant and all square submatrices of larger size have determinant zero. Use this fact to find the rank of

$$A = \begin{bmatrix} 3 & -1 & 3 & 2 & 5 \\ 5 & -3 & 2 & 3 & 4 \\ 1 & -3 & -5 & 0 & -7 \\ 7 & -5 & 1 & 4 & 1 \end{bmatrix}$$

Check your result by computing the rank of A in a different way.

Exercise Set 6.3

1. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^2.
   (a) $(0, 1)$, $(2, 0)$
   (b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
   (c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
   (d) $(0, 0)$, $(0, 1)$

2. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^3.
   (a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$
   (b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$
   (c) $(1, 0, 0)$, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(0, 0, 1)$
   (d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

3. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on P_2 (see Example 7 of Section 6.1).
   (a) $p_1(x) = \frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2$, $p_2(x) = \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2$, $p_3(x) = \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$
   (b) $p_1(x) = 1$, $p_2(x) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2$, $p_3(x) = x^2$

4. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on M_{22} (see Example 6 of Section 6.1).
   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

► In Exercises 5–6, show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space. ◄

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & 0 & -\frac{2}{3} \end{bmatrix}$

7. Verify that the vectors
$$\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right), \quad \mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right), \quad \mathbf{v}_3 = (0, 0, 1)$$
form an orthonormal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (1, -2, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

8. Use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in Exercise 7.

9. Verify that the vectors
$$\mathbf{v}_1 = (2, -2, 1), \quad \mathbf{v}_2 = (2, 1, -2), \quad \mathbf{v}_3 = (1, 2, 2)$$
form an orthogonal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (-1, 0, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

10. Verify that the vectors
$$\mathbf{v}_1 = (1, -1, 2, -1), \quad \mathbf{v}_2 = (-2, 2, 3, 2),$$
$$\mathbf{v}_3 = (1, 2, 0, -1), \quad \mathbf{v}_4 = (1, 0, 0, 1)$$
form an orthogonal basis for R^4 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (1, 1, 1, 1)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

► In Exercises 11–14, find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in the stated exercise. ◄

11. Exercise 7
12. Exercise 8
13. Exercise 9
14. Exercise 10

► In Exercises 15–18, let R^2 have the Euclidean inner product.

(a) Find the orthogonal projection of $\mathbf{u}$ onto the line spanned by the vector $\mathbf{v}$.

(b) Find the component of $\mathbf{u}$ orthogonal to the line spanned by the vector $\mathbf{v}$, and confirm that this component is orthogonal to the line. ◄

15. $\mathbf{u} = (-1, 6)$; $\mathbf{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$
16. $\mathbf{u} = (2, 3)$; $\mathbf{v} = \left(\frac{5}{13}, \frac{12}{13}\right)$
17. $\mathbf{u} = (2, 3)$; $\mathbf{v} = (1, 1)$
18. $\mathbf{u} = (3, -1)$; $\mathbf{v} = (3, 4)$

► In Exercises 19–22, let R^3 have the Euclidean inner product.

(a) Find the orthogonal projection of $\mathbf{u}$ onto the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Find the component of $\mathbf{u}$ orthogonal to the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and confirm that this component is orthogonal to the plane. ◄

19. $\mathbf{u} = (4, 2, 1)$; $\mathbf{v}_1 = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$, $\mathbf{v}_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$
20. $\mathbf{u} = (3, -1, 2)$; $\mathbf{v}_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
21. $\mathbf{u} = (1, 0, 3)$; $\mathbf{v}_1 = (1, -2, 1)$, $\mathbf{v}_2 = (2, 1, 0)$
22. $\mathbf{u} = (1, 0, 2)$; $\mathbf{v}_1 = (3, 1, 2)$, $\mathbf{v}_2 = (-1, 1, 1)$

► In Exercises 23–24, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -2)$ on the subspace W spanned by these vectors. ◄

23. $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (1, 1, -1, -1)$
24. $\mathbf{v}_1 = (0, 1, -4, -1)$, $\mathbf{v}_2 = (3, 5, 1, 1)$

In Exercises 25–26, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -1)$ onto the subspace W spanned by these vectors.

25. $\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$, $\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$

26. $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, $\mathbf{v}_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

In Exercises 27–28, let R^2 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis. Draw both sets of basis vectors in the xy-plane.

27. $\mathbf{u}_1 = (1, -3)$, $\mathbf{u}_2 = (2, 2)$

28. $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (3, -5)$

In Exercises 29–30, let R^3 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

29. $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (-1, 1, 0)$, $\mathbf{u}_3 = (1, 2, 1)$

30. $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (3, 7, -2)$, $\mathbf{u}_3 = (0, 4, 1)$

31. Let R^4 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0, 2, 1, 0), \quad \mathbf{u}_2 = (1, -1, 0, 0),$$
$$\mathbf{u}_3 = (1, 2, 0, -1), \quad \mathbf{u}_4 = (1, 0, 0, 1)$$

32. Let R^3 have the Euclidean inner product. Find an orthonormal basis for the subspace spanned by $(0, 1, 2)$, $(-1, 0, 1)$, $(-1, 1, 3)$.

33. Let $\mathbf{b}$ and W be as in Exercise 23. Find vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

34. Let $\mathbf{b}$ and W be as in Exercise 25. Find vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

35. Let R^3 have the Euclidean inner product. The subspace of R^3 spanned by the vectors $\mathbf{u}_1 = (1, 1, 1)$ and $\mathbf{u}_2 = (2, 0, -1)$ is a plane passing through the origin. Express $\mathbf{w} = (1, 2, 3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ lies in the plane and $\mathbf{w}_2$ is perpendicular to the plane.

36. Let R^4 have the Euclidean inner product. Express the vector $\mathbf{w} = (-1, 2, 6, 0)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in the space W spanned by $\mathbf{u}_1 = (-1, 0, 1, 2)$ and $\mathbf{u}_2 = (0, 1, 0, 1)$, and $\mathbf{w}_2$ is orthogonal to W.

37. Let R^3 have the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$$

Use the Gram–Schmidt process to transform $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (1, 0, 0)$ into an orthonormal basis.

38. Verify that the set of vectors $\{(1, 0), (0, 1)\}$ is orthogonal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1 v_1 + u_2 v_2$ on R^2; then convert it to an orthonormal set by normalizing the vectors.

39. Find vectors $\mathbf{x}$ and $\mathbf{y}$ in R^2 that are orthonormal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ but are not orthonormal with respect to the Euclidean inner product.

40. In Example 3 of Section 4.9 we found the orthogonal projection of the vector $\mathbf{x} = (1, 5)$ onto the line through the origin making an angle of $\pi/6$ radians with the positive x-axis. Solve that same problem using Theorem 6.3.4.

41. This exercise illustrates that the orthogonal projection resulting from Formula (12) in Theorem 6.3.4 does not depend on which orthogonal basis vectors are used.

(a) Let R^3 have the Euclidean inner product, and let W be the subspace of R^3 spanned by the orthogonal vectors

$$\mathbf{v}_1 = (1, 0, 1) \quad \text{and} \quad \mathbf{v}_2 = (0, 1, 0)$$

Show that the orthogonal vectors

$$\mathbf{v}_1' = (1, 1, 1) \quad \text{and} \quad \mathbf{v}_2' = (1, -2, 1)$$

span the same subspace W.

(b) Let $\mathbf{u} = (-3, 1, 7)$ and show that the same vector $\text{proj}_W \mathbf{u}$ results regardless of which of the bases in part (a) is used for its computation.

42. (*Calculus required*) Use Theorem 6.3.2(*a*) to express the following polynomials as linear combinations of the first three Legendre polynomials (see the Remark following Example 9).

(a) $1 + x + 4x^2$ (b) $2 - 7x^2$ (c) $4 + 3x$

43. (*Calculus required*) Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$$

Apply the Gram–Schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis.

44. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix}$$

In Exercises 45–48, we obtained the column vectors of Q by applying the Gram–Schmidt process to the column vectors of A. Find a QR-decomposition of the matrix A.

45. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

46. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$

49. Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

50. In the Remark following Example 8 we discussed two alternative ways to perform the calculations in the Gram–Schmidt process: normalizing each orthogonal basis vector as soon as it is calculated and scaling the orthogonal basis vectors at each step to eliminate fractions. Try these methods in Example 8.

Working with Proofs

51. Prove part (*a*) of Theorem 6.3.6.

52. In Step 3 of the proof of Theorem 6.3.5, it was stated that "the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ ensures that $\mathbf{v}_3 \neq \mathbf{0}$." Prove this statement.

53. Prove that the diagonal entries of R in Formula (16) are nonzero.

54. Show that matrix Q in Example 10 has the property $QQ^T = I_3$, and prove that every $m \times n$ matrix Q with orthonormal column vectors has the property $QQ^T = I_m$.

55. (a) Prove that if W is a subspace of a finite-dimensional vector space V, then the mapping $T: V \to W$ defined by $T(\mathbf{v}) = \text{proj}_W \mathbf{v}$ is a linear transformation.

    (b) What are the range and kernel of the transformation in part (a)?

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) Every linearly independent set of vectors in an inner product space is orthogonal.

(b) Every orthogonal set of vectors in an inner product space is linearly independent.

(c) Every nontrivial subspace of R^3 has an orthonormal basis with respect to the Euclidean inner product.

(d) Every nonzero finite-dimensional inner product space has an orthonormal basis.

(e) $\text{proj}_W \mathbf{x}$ is orthogonal to every vector of W.

(f) If A is an $n \times n$ matrix with a nonzero determinant, then A has a QR-decomposition.

Working with Technology

T1. (a) Use the Gram–Schmidt process to find an orthonormal basis relative to the Euclidean inner product for the column space of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

(b) Use the method of Example 9 to find a QR-decomposition of A.

T2. Let P_4 have the evaluation inner product at the points $-2, -1, 0, 1, 2$. Find an orthogonal basis for P_4 relative to this inner product by applying the Gram–Schmidt process to the vectors

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \quad \mathbf{p}_3 = x^3, \quad \mathbf{p}_4 = x^4$$

6.4 Best Approximation; Least Squares

There are many applications in which some linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns should be consistent on physical grounds but fails to be so because of measurement errors in the entries of A or $\mathbf{b}$. In such cases one looks for vectors that come as close as possible to being solutions in the sense that they minimize $\|\mathbf{b} - A\mathbf{x}\|$ with respect to the Euclidean inner product on R^m. In this section we will discuss methods for finding such minimizing vectors.

Least Squares Solutions of Linear Systems

Suppose that $A\mathbf{x} = \mathbf{b}$ is an *inconsistent* linear system of m equations in n unknowns in which we suspect the inconsistency to be caused by errors in the entries of A or $\mathbf{b}$. Since no exact solution is possible, we will look for a vector $\mathbf{x}$ that comes as "close as possible" to being a solution in the sense that it minimizes $\|\mathbf{b} - A\mathbf{x}\|$ with respect to the Euclidean

Exercise Set 6.4

In Exercises 1–2, find the associated normal equation.

1. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

In Exercises 3–6, find the least squares solution of the equation $A\mathbf{x} = \mathbf{b}$.

3. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}$

In Exercises 7–10, find the least squares error vector and least squares error of the stated equation. Verify that the least squares error vector is orthogonal to the column space of A.

7. The equation in Exercise 3.

8. The equation in Exercise 4.

9. The equation in Exercise 5.

10. The equation in Exercise 6.

In Exercises 11–14, find parametric equations for all least squares solutions of $A\mathbf{x} = \mathbf{b}$, and confirm that all of the solutions have the same error vector.

11. $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

13. $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

In Exercises 15–16, use Theorem 6.4.2 to find the orthogonal projection of $\mathbf{b}$ on the column space of A, and check your result using Theorem 6.4.4.

15. $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

16. $A = \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$

17. Find the orthogonal projection of $\mathbf{u}$ on the subspace of R^3 spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{u} = (1, -6, 1);\quad \mathbf{v}_1 = (-1, 2, 1),\quad \mathbf{v}_2 = (2, 2, 4)$$

18. Find the orthogonal projection of $\mathbf{u}$ on the subspace of R^4 spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{u} = (6, 3, 9, 6);\quad \mathbf{v}_1 = (2, 1, 1, 1),\quad \mathbf{v}_2 = (1, 0, 1, 1),$$
$$\mathbf{v}_3 = (-2, -1, 0, -1)$$

In Exercises 19–20, use the method of Example 3 to find the standard matrix for the orthogonal projection on the stated subspace of R^2. Compare your result to that in Table 3 of Section 4.9.

19. the x-axis

20. the y-axis

In Exercises 21–22, use the method of Example 3 to find the standard matrix for the orthogonal projection on the stated subspace of R^3. Compare your result to that in Table 4 of Section 4.9.

21. the xz-plane

22. the yz-plane

In Exercises 23–24, a QR-factorization of A is given. Use it to find the least squares solution of $A\mathbf{x} = \mathbf{b}$.

23. $A = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 5 & -\frac{1}{5} \\ 0 & \frac{7}{5} \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

24. $A = \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -10 \\ 0 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$

25. Let W be the plane with equation $5x - 3y + z = 0$.

(a) Find a basis for W.

(b) Find the standard matrix for the orthogonal projection onto W.

26. Let W be the line with parametric equations

$$x = 2t, \quad y = -t, \quad z = 4t$$

(a) Find a basis for W.

(b) Find the standard matrix for the orthogonal projection on W.

27. Find the orthogonal projection of $\mathbf{u} = (5, 6, 7, 2)$ on the solution space of the homogeneous linear system

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 0 \\ 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

28. Show that if $\mathbf{w} = (a, b, c)$ is a nonzero vector, then the standard matrix for the orthogonal projection of R^3 onto the line span$\{\mathbf{w}\}$ is

$$P = \frac{1}{a^2 + b^2 + c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

29. Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of R^n onto the row space of A.

Working with Proofs

30. Prove: If A has linearly independent column vectors, and if $A\mathbf{x} = \mathbf{b}$ is consistent, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ and the exact solution of $A\mathbf{x} = \mathbf{b}$ are the same.

31. Prove: If A has linearly independent column vectors, and if $\mathbf{b}$ is orthogonal to the column space of A, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{0}$.

32. Prove the implication $(b) \Rightarrow (a)$ of Theorem 6.4.3.

True-False Exercises

TF. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) If A is an $m \times n$ matrix, then A^TA is a square matrix.

(b) If A^TA is invertible, then A is invertible.

(c) If A is invertible, then A^TA is invertible.

(d) If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then $A^TA\mathbf{x} = A^T\mathbf{b}$ is also consistent.

(e) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent linear system, then $A^TA\mathbf{x} = A^T\mathbf{b}$ is also inconsistent.

(f) Every linear system has a least squares solution.

(g) Every linear system has a unique least squares solution.

(h) If A is an $m \times n$ matrix with linearly independent columns and $\mathbf{b}$ is in R^m, then $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.

Working with Technology

T1. (a) Use Theorem 6.4.4 to show that the following linear system has a unique least squares solution, and use the method of Example 1 to find it.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 2x_2 + x_3 &= 10 \\ 9x_1 + 3x_2 + x_3 &= 9 \\ 16x_1 + 4x_2 + x_3 &= 16 \end{aligned}$$

(b) Check your result in part (a) using Formula (9).

T2. Use your technology utility to perform the computations and confirm the results obtained in Example 2.

6.5 Mathematical Modeling Using Least Squares

In this section we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial curve to a set of experimentally determined points in the plane.

Fitting a Curve to Data

A common problem in experimental work is to obtain a mathematical relationship $y = f(x)$ between two variables x and y by "fitting" a curve to points in the plane corresponding to various experimentally determined values of x and y, say

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

On the basis of theoretical considerations or simply by observing the pattern of the points, the experimenter decides on the general form of the curve $y = f(x)$ to be fitted. This curve is called a ***mathematical model*** of the data. Some examples are (Figure 6.5.1):