Exercise Set 5.1

In Exercises 1–4, confirm by multiplication that $\mathbf{x}$ is an eigenvector of A, and find the corresponding eigenvalue.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In each part of Exercises 5–6, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

5. (a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

In Exercises 7–12, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

7. $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ 8. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ 10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

In Exercises 13–14, find the characteristic equation of the matrix by inspection.

13. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$ 14. $\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

In Exercises 15–16, find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula. [*Suggestion:* Work with the standard matrix for the operator.]

15. $T(x, y) = (x + 4y, 2x + 3y)$

16. $T(x, y, z) = (2x - y - z,\ x - z,\ -x + y + 2z)$

17. (***Calculus required***) Let $D^2: C^\infty(-\infty, \infty) \to C^\infty(-\infty, \infty)$ be the operator that maps a function into its second derivative.
   (a) Show that D^2 is linear.
   (b) Show that if ω is a positive constant, then $\sin\sqrt{\omega}x$ and $\cos\sqrt{\omega}x$ are eigenvectors of D^2, and find their corresponding eigenvalues.

18. (***Calculus required***) Let $D^2: C^\infty \to C^\infty$ be the linear operator in Exercise 17. Show that if ω is a positive constant, then $\sinh\sqrt{\omega}x$ and $\cosh\sqrt{\omega}x$ are eigenvectors of D^2, and find their corresponding eigenvalues.

In each part of Exercises 19–20, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on R^2. Refer to the tables in Section 4.9 and use geometric reasoning to find the answers. No computations are needed.

19. (a) Reflection about the line $y = x$.
   (b) Orthogonal projection onto the x-axis.
   (c) Rotation about the origin through a positive angle of 90°.
   (d) Contraction with factor k $(0 \le k < 1)$.
   (e) Shear in the x-direction by a factor k $(k \ne 0)$.

20. (a) Reflection about the y-axis.
   (b) Rotation about the origin through a positive angle of 180°.
   (c) Dilation with factor k $(k > 1)$.
   (d) Expansion in the y-direction with factor k $(k > 1)$.
   (e) Shear in the y-direction by a factor k $(k \ne 0)$.

In each part of Exercises 21–22, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on R^3. Refer to the tables in Section 4.9 and use geometric reasoning to find the answers. No computations are needed.

21. (a) Reflection about the xy-plane.
   (b) Orthogonal projection onto the xz-plane.
   (c) Counterclockwise rotation about the positive x-axis through an angle of 90°.
   (d) Contraction with factor k $(0 \le k < 1)$.

22. (a) Reflection about the xz-plane.
   (b) Orthogonal projection onto the yz-plane.
   (c) Counterclockwise rotation about the positive y-axis through an angle of 180°.
   (d) Dilation with factor k $(k > 1)$.

23. Let A be a 2×2 matrix, and call a line through the origin of R^2 ***invariant*** under A if $A\mathbf{x}$ lies on the line when $\mathbf{x}$ does. Find equations for all lines in R^2, if any, that are invariant under the given matrix.

(a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

24. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

(a) $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$

(b) $p(\lambda) = \lambda^4 - \lambda^3 + 7$

[*Hint:* See the proof of Theorem 5.1.4.]

25. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What is the size of A?

(b) Is A invertible?

(c) How many eigenspaces does A have?

26. The eigenvectors that we have been studying are sometimes called ***right eigenvectors*** to distinguish them from ***left eigenvectors***, which are $n \times 1$ column matrices $\mathbf{x}$ that satisfy the equation $\mathbf{x}^T A = \mu \mathbf{x}^T$ for some scalar μ. For a given matrix A, how are the right eigenvectors and their corresponding eigenvalues related to the left eigenvectors and their corresponding eigenvalues?

27. Find a 3×3 matrix A that has eigenvalues 1, -1, and 0, and for which

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

are their corresponding eigenvectors.

Working with Proofs

28. Prove that the characteristic equation of a 2×2 matrix A can be expressed as $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$, where $\operatorname{tr}(A)$ is the trace of A.

29. Use the result in Exercise 28 to show that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the solutions of the characteristic equation of A are

$$\lambda = \tfrac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\,\right]$$

Use this result to show that A has

(a) two distinct real eigenvalues if $(a-d)^2 + 4bc > 0$.

(b) two repeated real eigenvalues if $(a-d)^2 + 4bc = 0$.

(c) complex conjugate eigenvalues if $(a-d)^2 + 4bc < 0$.

30. Let A be the matrix in Exercise 29. Show that if $b \neq 0$, then

$$\mathbf{x}_1 = \begin{bmatrix} -b \\ a - \lambda_1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -b \\ a - \lambda_2 \end{bmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \tfrac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\,\right]$$

and

$$\lambda_2 = \tfrac{1}{2}\left[(a+d) - \sqrt{(a-d)^2 + 4bc}\,\right]$$

31. Use the result of Exercise 28 to prove that if

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2$$

is the characteristic polynomial of a 2×2 matrix, then

$$p(A) = A^2 + c_1 A + c_2 I = 0$$

(Stated informally, A satisfies its characteristic equation. This result is true as well for $n \times n$ matrices.)

32. Prove: If a, b, c, and d are integers such that $a + b = c + d$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has integer eigenvalues.

33. Prove: If λ is an eigenvalue of an invertible matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector.

34. Prove: If λ is an eigenvalue of A, $\mathbf{x}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and $\mathbf{x}$ is a corresponding eigenvector.

35. Prove: If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s and $\mathbf{x}$ is a corresponding eigenvector.

36. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

and then use Exercises 33 and 34 to find the eigenvalues and bases for the eigenspaces of

(a) A^{-1} (b) $A - 3I$ (c) $A + 2I$

37. Prove that the characteristic polynomial of an $n \times n$ matrix A has degree n and that the coefficient of λ^n in that polynomial is 1.

38. (a) Prove that if A is a square matrix, then A and A^T have the same eigenvalues. [*Hint:* Look at the characteristic equation $\det(\lambda I - A) = 0$.]

(b) Show that A and A^T need not have the same eigenspaces. [*Hint:* Use the result in Exercise 30 to find a 2×2 matrix for which A and A^T have different eigenspaces.]

39. Prove that the intersection of any two distinct eigenspaces of a matrix A is $\{\mathbf{0}\}$.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If A is a square matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero scalar λ, then $\mathbf{x}$ is an eigenvector of A.

(b) If λ is an eigenvalue of a matrix A, then the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible.

(d) If λ is an eigenvalue of a matrix A, then the eigenspace of A corresponding to λ is the set of eigenvectors of A corresponding to λ.

(e) The eigenvalues of a matrix A are the same as the eigenvalues of the reduced row echelon form of A.

(f) If 0 is an eigenvalue of a matrix A, then the set of columns of A is linearly independent.

Working with Technology

T1. For the given matrix A, find the characteristic polynomial and the eigenvalues, and then use the method of Example 7 to find bases for the eigenspaces.

$$A = \begin{bmatrix} -8 & 33 & 38 & 173 & -30 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & -5 & -25 & 1 \\ 0 & 0 & 1 & 5 & 0 \\ 4 & -16 & -19 & -86 & 15 \end{bmatrix}$$

T2. The Cayley–Hamilton Theorem states that every square matrix satisfies its characteristic equation; that is, if A is an $n \times n$ matrix whose characteristic equation is

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

then $A^n + c_1 A^{n-1} + \cdots + c_n = 0$.

(a) Verify the Cayley–Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

(b) Use the result in Exercise 28 to prove the Cayley–Hamilton Theorem for 2×2 matrices.

5.2 Diagonalization

In this section we will be concerned with the problem of finding a basis for R^n that consists of eigenvectors of an $n \times n$ matrix A. Such bases can be used to study geometric properties of A and to simplify various numerical computations. These bases are also of physical significance in a wide variety of applications, some of which will be considered later in this text.

The Matrix Diagonalization Problem

Products of the form $P^{-1}AP$ in which A and P are $n \times n$ matrices and P is invertible will be our main topic of study in this section. There are various ways to think about such products, one of which is to view them as transformations

$$A \to P^{-1}AP$$

in which the matrix A is mapped into the matrix $P^{-1}AP$. These are called ***similarity transformations***. Such transformations are important because they preserve many properties of the matrix A. For example, if we let $B = P^{-1}AP$, then A and B have the same determinant since

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$$

Since the eigenvector $\mathbf{v}_{r+1}$ is nonzero, it follows that

$$c_{r+1} = 0 \tag{8}$$

But equations (7) and (8) contradict the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero so the proof is complete. ◀

Exercise Set 5.2

▶ In Exercises 1–4, show that A and B are not similar matrices. ◀

1. $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

▶ In Exercises 5–8, find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$. ◀

5. $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

6. $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

   (a) Find the eigenvalues of A.

   (b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

   (c) Is A diagonalizable? Justify your conclusion.

10. Follow the directions in Exercise 9 for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

▶ In Exercises 11–14, find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A, and find $P^{-1}AP$. ◀

11. $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

14. $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

▶ In each part of Exercises 15–16, the characteristic equation of a matrix A is given. Find the size of the matrix and the possible dimensions of its eigenspaces. ◀

15. (a) $(\lambda - 1)(\lambda + 3)(\lambda - 5) = 0$

    (b) $\lambda^2(\lambda - 1)(\lambda - 2)^3 = 0$

16. (a) $\lambda^3(\lambda^2 - 5\lambda - 6) = 0$

    (b) $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

▶ In Exercises 17–18, use the method of Example 6 to compute the matrix A^{10}. ◀

17. $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

19. Let

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

   Confirm that P diagonalizes A, and then compute A^{11}.

20. Let

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

   Confirm that P diagonalizes A, and then compute each of the following powers of A.

   (a) A^{1000} (b) A^{-1000} (c) A^{2301} (d) A^{-2301}

21. Find A^n if n is a positive integer and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

22. Show that the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are similar.

23. We know from Table 1 that similar matrices have the same rank. Show that the converse is false by showing that the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have the same rank but are not similar. [*Suggestion:* If they were similar, then there would be an invertible 2×2 matrix P for which $AP = PB$. Show that there is no such matrix.]

24. We know from Table 1 that similar matrices have the same eigenvalues. Use the method of Exercise 23 to show that the converse is false by showing that the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same eigenvalues but are not similar.

25. If A, B, and C are $n \times n$ matrices such that A is similar to B and B is similar to C, do you think that A must be similar to C? Justify your answer.

26. (a) Is it possible for an $n \times n$ matrix to be similar to itself? Justify your answer.

(b) What can you say about an $n \times n$ matrix that is similar to $0_{n \times n}$? Justify your answer.

(c) Is is possible for a nonsingular matrix to be similar to a singular matrix? Justify your answer.

27. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What can you say about the dimensions of the eigenspaces of A?

(b) What can you say about the dimensions of the eigenspaces if you know that A is diagonalizable?

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of eigenvectors of A, all of which correspond to the same eigenvalue of A, what can you say about that eigenvalue?

28. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that

(a) A is diagonalizable if $(a - d)^2 + 4bc > 0$.

(b) A is not diagonalizable if $(a - d)^2 + 4bc < 0$.

[*Hint:* See Exercise 29 of Section 5.1.]

29. In the case where the matrix A in Exercise 28 is diagonalizable, find a matrix P that diagonalizes A. [*Hint:* See Exercise 30 of Section 5.1.]

In Exercises 30–33, find the standard matrix A for the given linear operator, and determine whether that matrix is diagonalizable. If diagonalizable, find a matrix P that diagonalizes A.

30. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

31. $T(x_1, x_2) = (-x_2, -x_1)$

32. $T(x_1, x_2, x_3) = (8x_1 + 3x_2 - 4x_3, -3x_1 + x_2 + 3x_3, 4x_1 + 3x_2)$

33. $T(x_1, x_2, x_3) = (3x_1, x_2, x_1 - x_2)$

34. If P is a fixed $n \times n$ matrix, then the similarity transformation

$$A \rightarrow P^{-1}AP$$

can be viewed as an operator $S_P(A) = P^{-1}AP$ on the vector space M_{nn} of $n \times n$ matrices.

(a) Show that S_P is a linear operator.

(b) Find the kernel of S_P.

(c) Find the rank of S_P.

Working with Proofs

35. Prove that similar matrices have the same rank and nullity.

36. Prove that similar matrices have the same trace.

37. Prove that if A is diagonalizable, then so is A^k for every positive integer k.

38. We know from Table 1 that similar matrices, A and B, have the same eigenvalues. However, it is not true that those eigenvalues have the same corresponding eigenvectors for the two matrices. Prove that if $B = P^{-1}AP$, and $\mathbf{v}$ is an eigenvector of B corresponding to the eigenvalue λ, then $P\mathbf{v}$ is the eigenvector of A corresponding to λ.

39. Let A be an $n \times n$ matrix, and let $q(A)$ be the matrix

$$q(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n$$

(a) Prove that if $B = P^{-1}AP$, then $q(B) = P^{-1}q(A)P$.

(b) Prove that if A is diagonalizable, then so is $q(A)$.

40. Prove that if A is a diagonalizable matrix, then the rank of A is the number of nonzero eigenvalues of A.

41. This problem will lead you through a proof of the fact that the algebraic multiplicity of an eigenvalue of an $n \times n$ matrix A is greater than or equal to the geometric multiplicity. For this purpose, assume that λ_0 is an eigenvalue with geometric multiplicity k.

(a) Prove that there is a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for R^n in which the first k vectors of B form a basis for the eigenspace corresponding to λ_0.

(b) Let P be the matrix having the vectors in B as columns. Prove that the product AP can be expressed as

$$AP = P\begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

[*Hint:* Compare the first k column vectors on both sides.]

(c) Use the result in part (b) to prove that A is similar to

$$C = \begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

and hence that A and C have the same characteristic polynomial.

(d) By considering $\det(\lambda I - C)$, prove that the characteristic polynomial of C (and hence A) contains the factor $(\lambda - \lambda_0)$ at least k times, thereby proving that the algebraic multiplicity of λ_0 is greater than or equal to the geometric multiplicity k.

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) An $n \times n$ matrix with fewer than n distinct eigenvalues is not diagonalizable.

(b) An $n \times n$ matrix with fewer than n linearly independent eigenvectors is not diagonalizable.

(c) If A and B are similar $n \times n$ matrices, then there exists an invertible $n \times n$ matrix P such that $PA = BP$.

(d) If A is diagonalizable, then there is a unique matrix P such that $P^{-1}AP$ is diagonal.

(e) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

(f) If A is diagonalizable, then A^T is diagonalizable.

(g) If there is a basis for R^n consisting of eigenvectors of an $n \times n$ matrix A, then A is diagonalizable.

(h) If every eigenvalue of a matrix A has algebraic multiplicity 1, then A is diagonalizable.

(i) If 0 is an eigenvalue of a matrix A, then A^2 is singular.

Working with Technology

T1. Generate a random 4×4 matrix A and an invertible 4×4 matrix P and then confirm, as stated in Table 1, that $P^{-1}AP$ and A have the same

(a) determinant.

(b) rank.

(c) nullity.

(d) trace.

(e) characteristic polynomial.

(f) eigenvalues.

T2. (a) Use Theorem 5.2.1 to show that the following matrix is diagonalizable.

$$A = \begin{bmatrix} -13 & -60 & -60 \\ 10 & 42 & 40 \\ -5 & -20 & -18 \end{bmatrix}$$

(b) Find a matrix P that diagonalizes A.

(c) Use the method of Example 6 to compute A^{10}, and check your result by computing A^{10} directly.

T3. Use Theorem 5.2.1 to show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} -10 & 11 & -6 \\ -15 & 16 & -10 \\ -3 & 3 & -2 \end{bmatrix}$$

5.3 Complex Vector Spaces

Because the characteristic equation of any square matrix can have complex solutions, the notions of complex eigenvalues and eigenvectors arise naturally, even within the context of matrices with real entries. In this section we will discuss this idea and apply our results to study symmetric matrices in more detail. A review of the essentials of complex numbers appears in the back of this text.

Review of Complex Numbers

Recall that if $z = a + bi$ is a complex number, then:

- $\text{Re}(z) = a$ and $\text{Im}(z) = b$ are called the ***real part*** of z and the ***imaginary part*** of z, respectively,
- $|z| = \sqrt{a^2 + b^2}$ is called the ***modulus*** (or ***absolute value***) of z,
- $\overline{z} = a - bi$ is called the ***complex conjugate*** of z,