

Asymptotically self-similar shock formation for 1d fractal Burgers equation

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Abstract

For $0 < \alpha < 1/3$ we construct unique solutions to the fractal Burgers equation

$$\partial_t u + u \partial_x u + (-\Delta)^\alpha u = 0$$

which develop a first shock in finite time, starting from smooth generic initial data. This first singularity is an asymptotically self-similar, stable H^6 perturbation of a stable, self-similar Burgers shock profile. Furthermore, we are able to compute the spatio-temporal location and Hölder regularity for the first singularity. There are many results showing that gradient blowup occurs in finite time for the supercritical range, but the present result is the first example where singular solutions have been explicitly constructed and so precisely characterized.

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1 Introduction

In this paper we study *fractal Burgers equation*

$$\begin{aligned} \partial_t u + u \partial_x u + (-\Delta)^\alpha u &= 0, & \text{on } \mathbf{R} \times \mathbf{R} \\ u(x, 0) &= u_0(x), & \text{on } \mathbf{R} \end{aligned} \quad (1.1)$$

where the fractional Laplacian $(-\Delta)^\alpha$ is defined as the Fourier multiplier $|\xi|^{2\alpha}$, and can be represented by the Calderon-Zygmund singular integral

$$((-\Delta)^\alpha u)(x) = C_\alpha \int_{\mathbf{R}} \frac{u(x) - u(\eta)}{|x - \eta|^{2\alpha+1}} d\eta, \quad C_\alpha = \frac{4^\alpha \Gamma(\alpha + 1/2)}{\pi^{1/2} |\Gamma(-\alpha)|}, \quad (1.2)$$

taken in the principle value sense. For the remainder of this paper we will assume all singular integrals are taken in a principle value sense. The formula (1.2) is well known and a derivation for any dimension is given in [CC04].

Equation (1.1) models the competing effects of viscous regularization and a Burgers-type nonlinearity [ADV07, KNS08]. The case $\alpha = 0$ corresponds to Burger’s equation, which forms shocks in finite time from smooth initial data, while $\alpha = 1$ corresponds to viscous Burger’s equation, whose solutions immediately regularize to C^∞ [Whi11]. In particular, (1.1) is a dramatically simplified model of the interaction between viscous regularization and nonlinearity in the Navier-Stokes equations.

The equation (1.1) has critical exponent $\alpha = 1/2$, at which the orders of the regularization and the nonlinearity match. It is known that equation (1.1) forms gradient discontinuities in the supercritical case $0 < \alpha < 1/2$, and remains globally smooth from smooth data in both the critical case $\alpha = 1/2$ and the subcritical case $1/2 < \alpha < 1$ [KNS08, ADV07].

1.1 Historical Study of Fractal Burgers Equation

Equation (1.1) has been well-studied in the last two decades. Biler, Funaki, and Woyczynski [BFW98] established the existence of self-similar solutions, L^2 energy estimates, and global well-posedness in H^1 for the range $3/4 < \alpha \leq 1$, among other things. More complete studies of the existence and blowup of solutions to (1.1) were carried out independently by Alibaud, Droniou, and Vovelle [ADV07], and Kiselev, Nazarov, and Shterenberg [KNS08]. Although they used very different methods, both proved finite-time blowup in the supercritical case, and global existence in the critical and subcritical cases. Dong, Du, and Li [DDL09] produced an explicit open set of initial data which form shocks in finite time. Albritton and Beekie have examined the long term asymptotic behavior of solutions to (1.1) in the critical case $\alpha = 1/2$ and showed that solutions asymptotically approach a self-similar solution [AB20]. In this paper, we produce an open set of initial data which asymptotically approach a self-similar solution to Burger’s equation, and characterize the Hölder regularity of the shock profile.

As this paper was being submitted to the arXiv we were made aware of an independent work by Oh and Pasqualotto which proves the same result using different techniques [OP21].

1.2 Similarity Variables, Modulation, and Singularity Formation

The utility of similarity variables has been known for decades, and the comprehensive study of such methods was started by Barenblatt and Zel’Dovich [BZ72, Bar96] in the 20th century. Giga and Kohn [GK85, GK87] adapted these methods for studying singularity formation in the nonlinear heat equation, and made significant contributions to our understanding of the role of self-similarity in singular behavior. Eggers and Fontelos [EF08, EF15] have distilled the work of Barenblatt, Zel’Dovich, Giga, Kohn, and others into a digestible framework suitable for applications in many physical and analytic contexts. Most recently, Collot, Ghoul, and Masmoudi [CGM18] proved that shocks for Burgers equation are self-similar to leading order.

The present result is inspired by the results obtained by Buckmaster, Shkoller, and Vicol in their series of papers [BSV19a, BSV19b, BSV19c] which prove shock formation for the 2D isentropic compressible Euler, 3D isentropic compressible Euler, and 3D non-isentropic compressible Euler equations respectively. The fundamental idea underlying the method of Buckmaster, Shkoller, and Vicol is that instabilities in the evolution correspond to symmetries of the equation under study. Informally speaking, we introduce “modulation functions” which keep track of the equivalence class of the solution as it evolves.

1.3 Summary of Present Result

The present result shows that for a certain range of α , we can view the evolution (1.1) as a perturbation of *stable* Burgers shock dynamics:

Theorem 1.1. (*Imprecise Statement of Result*) *Starting from non-degenerate initial data, the solution of (1.1) forms a generic point shock. Furthermore, we can compute the time, location, and regularity of this shock. Furthermore, the shock is an asymptotically self-similar Burger's shock.*

The formal statement of this result is contained in Theorem 2.1 and Corollary 2.1.1. Our proof closely follows the strategy laid out in the recent works [BSV19a, BSV19b, BSV19c, BI20, Yan20], namely using modulated self-similar variables, careful bounds on Lagrangian trajectories, and a bootstrapping argument to prove the result.

Symmetry and Modulation. To illustrate the need for modulation, suppose we have an exact initial data $u(\cdot, 0)$ which we know leads to a shock, and in the course of our proof we use the fact that $u(0, 0) = 0$. We want to show that in some open neighborhood of $u(\cdot, 0)$ a shock will still form. In our case, we work with the H^6 topology, in which a generic perturbation of $u(\cdot, 0)$ no longer fixes the origin.

The reader can check that Fractal Burgers equation (1.1) is invariant under the following four parameter symmetry group

$$u_*(x, t) = \frac{1}{\lambda} u\left(\frac{x - x_0}{\lambda}, \frac{t - t_0}{\lambda}\right) + u_0, \quad (1.3)$$

where u solves (1.1).

Our solution to this problem is to track certain invariants which fix the equivalence class under the relevant symmetries. For instance, when we perturb $u(\cdot, 0)$ by a sufficiently small amount to $v(\cdot, 0)$, there is a small function $\xi(t)$ such that $v(\xi(t), t) = 0$ for small t . By keeping track of ξ , we fix the equivalence class of $v(\cdot, 0)$ under spatial translation. The modulation functions fix the equivalence class of the solution and allow us to drop the odd symmetry assumptions present in [ADV07] and prove shock formation on an open set in a strong topology.

2 Self-Similar Transformation, Initial Data, and Statement of Theorem

Our solutions will develop a gradient discontinuity at an *a priori* unknown time T_* and location x_* . Throughout ε denotes a small parameter which will be fixed during our proof.

2.1 The Self-Similar Transformation

We define the modulation variables $\tau, \xi, \kappa : [-\varepsilon, T_*] \rightarrow \mathbf{R}$, which respectively control the temporal location, spatial location, and density of the developing shock. We fix their initial values at time $t = -\varepsilon$ to be

$$\tau(-\varepsilon) = 0, \quad \xi(-\varepsilon) = 0, \quad \kappa(-\varepsilon) = \kappa_0. \quad (2.1)$$

By their definition, the modulation variables will satisfy $\tau(T_*) = T_*$ and $\xi(T_*) = x_*$, meaning that the gradient blowup occurs at the unique fixed point of τ .

We make the change to *modulated self-similar* variables described by

$$y(x, t) = \frac{x - \xi(t)}{(\tau(t) - t)^{3/2}}, \quad s(t) = -\log(\tau(t) - t) \quad (2.2)$$

and the modulated self-similar variables ansatz

$$u(x, t) = e^{-\frac{s}{2}} W(y, s) + \kappa(t). \quad (2.3)$$

This ansatz corresponds to the stable self-similar scaling for the stable Burgers profile Ψ (c.f. Appendix B). This scaling is a natural choice since we are considering the dynamics of (1.1) as a perturbation from the stable Burgers dynamics.

Plugging the ansatz (2.3) into (1.1) we obtain the PDE

$$\left(\partial_s - \frac{1}{2}\right) W + \left(\beta_\tau(W + e^{\frac{s}{2}}(\kappa - \dot{\xi})) + \frac{3}{2}y\right) \partial_x W = -\beta_\tau e^{-\frac{s}{2}} \dot{\kappa} - \beta_\tau e^{(3\alpha-1)s} (-\Delta)^\alpha W \quad (2.4)$$

for W in y and s and where

$$\beta_\tau := \frac{1}{1 - \dot{\tau}}.$$

For convenience we define the transport speed

$$g_W := \beta_\tau \left(W + e^{\frac{3}{2}}(\kappa - \dot{\xi}) \right) + \frac{3}{2}y. \quad (2.5)$$

The equations governing the evolution of the derivatives of W are recorded in Appendix C.

2.1.1 Modulation Functions and Constraints on the Evolution

Imposing the following constraints at $y = 0$ fully characterizes the developing shock, as well as fixing our choice of τ , ξ and κ :

$$W(0, s) = 0, \quad \partial_y W(0, s) = -1, \quad \partial_y^2 W(0, s) = 0. \quad (2.6)$$

For any function φ we denote $\varphi(0, s) =: \varphi^0(s) = \varphi^0$ out of convenience. We record the following identities for τ and ξ

$$\dot{\tau} = -e^{(3\alpha-1)s} ((-\Delta)^\alpha \partial_y W)^0(s), \quad (2.7a)$$

$$\kappa - \dot{\xi} = -e^{-s} \dot{\kappa} - e^{(3\alpha-\frac{3}{2})s} ((-\Delta)^\alpha W)^0(s) = -\frac{e^{(3\alpha-\frac{3}{2})s}}{\partial_y^3 W^0(s)} ((-\Delta)^\alpha \partial_y^2 W)^0(s). \quad (2.7b)$$

We compute (2.7a) by evaluating (C.1) at $y = 0$. Equation (2.7b) follows from evaluating (C.2) and (2.4) at $y = 0$.

2.2 Assumptions on the Initial Data

Let $M > 1$ be large, $\varepsilon < 1$ small. These constants will be fixed in the course of the proof and are independent of one another. This ensures that that we can make quantities of the form $M^p \varepsilon$, $p > 0$, arbitrarily small by taking ε sufficiently small. We use this frequently in what follows. We define the constants:

$$m = \frac{3}{16}(1 - 3\alpha), \quad \ell = \frac{7}{8}(1 - 3\alpha) \quad q = \min\left\{\frac{2}{3}(1 - 3\alpha), \frac{1}{6}\right\}, \quad (2.8)$$

which are chosen to satisfy certain constraints arising during the course of the bootstrap argument. We use the notation $\langle x \rangle := (1 + x^2)^{1/2}$ throughout and set

$$s_0 = -\log(\varepsilon) \quad \text{and} \quad h = (\log M)^{-2}. \quad (2.9)$$

Note that h depends on M but not on ε .

We break \mathbf{R} into three regions. We call $\mathbf{B}_h(0)$ the *Taylor region* since we will rely on Taylor expanding to close our bootstraps here. The next region is the time dependent annulus $\mathbf{B}_{e^{ms}}(0) \setminus \mathbf{B}_h(0)$ with m defined in (2.8). Finally we have the *far field* which is simply $\mathbf{R} \setminus \mathbf{B}_{e^{ms}}(0)$. Since the fractal Burgers equation does not propagate compact support, we must maintain quantitative control on the solution in this region.

2.2.1 Initial Data in Self-Similar Variables

We choose initial data $W(0, s)$ in self-similar variables satisfying the following pointwise equality at the origin:

$$W^0(s_0) = 0, \quad \partial_y W^0(s_0) = -\|\partial_y W(\cdot, s_0)\|_{L^\infty} = -1, \quad \partial_y^2 W^0(s_0) = 0. \quad (2.10)$$

We define $\widetilde{W} := W - \Psi$, where Ψ is the exact, self-similar Burgers profile (B.4). In this notation we assume

$$|\partial_y^3 \widetilde{W}^0(s_0)| \leq 6\varepsilon^{\frac{16}{9}(1-3\alpha)}, \quad (2.11)$$

$$|\widetilde{W}(y, s_0)| \leq \begin{cases} \varepsilon^{1-3\alpha}, & 0 \leq |y| \leq h \\ \frac{1}{2}\varepsilon^q \langle y \rangle^{1/3}, & h \leq |y| < \infty, \end{cases} \quad (2.12a)$$

$$(2.12b)$$

and

$$|\partial_y \widetilde{W}(y, s_0)| \leq \begin{cases} \varepsilon^{1-3\alpha}, & 0 \leq |y| \leq h \\ \frac{3}{4} \varepsilon^{\frac{3}{4} \ell} \langle y \rangle^{-2/3}, & h \leq |y| \leq \varepsilon^{-m} \\ |\partial_y \widetilde{W}(y, s_0)| \leq 2\varepsilon^2, & \varepsilon^{-m} \leq |y|. \end{cases} \quad (2.13a)$$

$$|\partial_y \widetilde{W}(y, s_0)| \leq \begin{cases} \varepsilon^{1-3\alpha}, & 0 \leq |y| \leq h \\ \frac{3}{4} \varepsilon^{\frac{3}{4} \ell} \langle y \rangle^{-2/3}, & h \leq |y| \leq \varepsilon^{-m} \\ |\partial_y \widetilde{W}(y, s_0)| \leq 2\varepsilon^2, & \varepsilon^{-m} \leq |y|. \end{cases} \quad (2.13b)$$

$$|\partial_y \widetilde{W}(y, s_0)| \leq \begin{cases} \varepsilon^{1-3\alpha}, & 0 \leq |y| \leq h \\ \frac{3}{4} \varepsilon^{\frac{3}{4} \ell} \langle y \rangle^{-2/3}, & h \leq |y| \leq \varepsilon^{-m} \\ |\partial_y \widetilde{W}(y, s_0)| \leq 2\varepsilon^2, & \varepsilon^{-m} \leq |y|. \end{cases} \quad (2.13c)$$

as well as the following bounds on the higher derivatives in the Taylor region $0 \leq |y| \leq h$

$$|\partial_y^2 \widetilde{W}(y, s_0)| \leq \varepsilon^{1-3\alpha}, \quad |\partial_y^3 \widetilde{W}(y, s_0)| \leq \varepsilon^{1-3\alpha}, \quad |\partial_y^4 \widetilde{W}(y, s_0)| \leq \varepsilon^{1-3\alpha}. \quad (2.14)$$

We will assume the norm bounds

$$\|\partial_y W(\cdot, s_0)\|_{L^2} \leq 200, \quad \|\partial_y^3 W(\cdot, s_0)\|_{L^\infty} \leq \frac{M^{\frac{4}{7}}}{2}, \quad \|\partial_y^6 W(\cdot, s_0)\|_{L^2} \leq \frac{M^{1/2}}{2}. \quad (2.15)$$

2.2.2 Initial Data in Physical Variables

We translate (2.10)-(2.15) into the physical variable. Our pointwise equalities at the origin become

$$u_0(0) = \kappa_0, \quad u'_0(0) = -\varepsilon^{-1} = -\|u'_0\|_{L^\infty}, \quad u''_0(0) = 0. \quad (2.16)$$

Inequality (2.11) becomes

$$|\partial_\theta^3 u_0(0) - 6\varepsilon^{-4}| \leq 6\varepsilon^{-4/9(5+12\alpha)}. \quad (2.17)$$

The initial bounds (2.12)-(2.13) in the self-similar variables become

$$|u(x, -\varepsilon)| \leq \begin{cases} \varepsilon^{\frac{1}{2}} \left(\Psi(x\varepsilon^{-\frac{3}{2}}) + \varepsilon^{1+3\alpha} \right) + \kappa_0, & 0 \leq |x| \leq h\varepsilon^{\frac{3}{2}} \\ \varepsilon^{\frac{1}{2}} \left(\Psi(x\varepsilon^{-\frac{3}{2}}) + \frac{1}{2} \varepsilon^q \langle x\varepsilon^{-\frac{3}{2}} \rangle^{1/3} \right) + \kappa_0, & h\varepsilon^{\frac{3}{2}} \leq |x| < \infty \end{cases} \quad (2.18a)$$

$$|u(x, -\varepsilon)| \leq \begin{cases} \varepsilon^{\frac{1}{2}} \left(\Psi(x\varepsilon^{-\frac{3}{2}}) + \frac{1}{2} \varepsilon^q \langle x\varepsilon^{-\frac{3}{2}} \rangle^{1/3} \right) + \kappa_0, & h\varepsilon^{\frac{3}{2}} \leq |x| < \infty \end{cases} \quad (2.18b)$$

$$|\partial_x u(x, -\varepsilon)| \leq \begin{cases} \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \varepsilon^{1-3\alpha} \right), & 0 \leq |x| \leq h\varepsilon^{\frac{3}{2}} \\ \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \frac{3}{4} \varepsilon^{\frac{3}{4} \ell} \langle x\varepsilon^{-\frac{3}{2}} \rangle^{-2/3} \right), & h\varepsilon^{\frac{3}{2}} \leq |x| \leq \varepsilon^{-m+\frac{3}{2}} \\ \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \frac{1}{2} \varepsilon^{1/2} 2\varepsilon \right), & \varepsilon^{-m+\frac{3}{2}} \leq |x| < \infty \end{cases} \quad (2.19a)$$

$$|\partial_x u(x, -\varepsilon)| \leq \begin{cases} \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \frac{3}{4} \varepsilon^{\frac{3}{4} \ell} \langle x\varepsilon^{-\frac{3}{2}} \rangle^{-2/3} \right), & h\varepsilon^{\frac{3}{2}} \leq |x| \leq \varepsilon^{-m+\frac{3}{2}} \\ \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \frac{1}{2} \varepsilon^{1/2} 2\varepsilon \right), & \varepsilon^{-m+\frac{3}{2}} \leq |x| < \infty \end{cases} \quad (2.19b)$$

$$|\partial_x u(x, -\varepsilon)| \leq \begin{cases} \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \varepsilon^{1-3\alpha} \right), & 0 \leq |x| \leq h\varepsilon^{\frac{3}{2}} \\ \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \frac{3}{4} \varepsilon^{\frac{3}{4} \ell} \langle x\varepsilon^{-\frac{3}{2}} \rangle^{-2/3} \right), & h\varepsilon^{\frac{3}{2}} \leq |x| \leq \varepsilon^{-m+\frac{3}{2}} \\ \varepsilon^{-1} \left(\Psi'(x\varepsilon^{-\frac{3}{2}}) + \frac{1}{2} \varepsilon^{1/2} 2\varepsilon \right), & \varepsilon^{-m+\frac{3}{2}} \leq |x| < \infty \end{cases} \quad (2.19c)$$

The higher derivative bounds (2.14) translate to

$$\begin{aligned} \partial_x^2 u(x, -\varepsilon) &\leq \varepsilon^{-5/2} \left(\varepsilon^{1-3\alpha} + \Psi^{(2)}(x\varepsilon^{-\frac{3}{2}}) \right), \\ \partial_x^3 u(x, -\varepsilon) &\leq \varepsilon^{-4} \left(\varepsilon^{1-3\alpha} + \Psi^{(3)}(x\varepsilon^{-\frac{3}{2}}) \right), \\ \partial_x^4 u(x, -\varepsilon) &\leq \varepsilon^{-11/2} \left(\varepsilon^{1-3\alpha} + \Psi^{(4)}(x\varepsilon^{-\frac{3}{2}}) \right), \end{aligned} \quad (2.20)$$

which are valid in the region $0 \leq |x| \leq h\varepsilon^{\frac{3}{2}}$. The norm bounds (2.15) become

$$\|\partial_x u(\cdot, -\varepsilon)\|_{L^2}^2 = \varepsilon^{-\frac{1}{2}} \|\partial_x W(\cdot, s_0)\|_{L^2}^2 \leq 40000 \varepsilon^{-\frac{1}{2}}, \quad (2.21a)$$

$$\|\partial_x^3 u(\cdot, -\varepsilon)\|_{L^\infty} \leq \varepsilon^{-4} \|\partial_x^3 W_0\|_{L^\infty} = \frac{\varepsilon^{-4}}{2} M^{\frac{4}{7}}, \quad (2.21b)$$

$$\|\partial_x^6 u(\cdot, -\varepsilon)\|_{L^2} = \varepsilon^{-4} \|\partial_x^6 W_0\|_{L^2} \leq \frac{\varepsilon^{-\frac{31}{4}}}{2} M^{1/2}. \quad (2.21c)$$

2.3 Main Theorem

We are now equipped to state the main result of this work

Theorem 2.1. *There exist constants $M, \varepsilon > 0$ sufficiently large and small respectively such that for some $u_0 \in H^6(\mathbf{R})$, with compact support, satisfying (2.18)-(2.21a) at $t = -\varepsilon$, the unique solution to (1.1) forms a shock in finite time. Furthermore*

- (i) For any $\bar{T} < T_*$, $u \in C([-\varepsilon, \bar{T}]; C^5(\mathbf{R}))$.
- (ii) The blowup location $\xi(T_*) = x_*$ is unique.
- (iii) The blowup time T_* occurs prior to $t = \frac{3}{4}\varepsilon^{\frac{8}{9}(1-3\alpha)}$ and the blowup location x_* satisfies $|x_*| \leq 4M\varepsilon$. The blowup time and location are explicitly computable.
- (iv) $\lim_{t \rightarrow T_*} \partial_\theta u = -\infty$ and furthermore we have the precise control

$$\frac{1}{2(T_* - t)} \leq |\partial_\theta u(\xi(t), t)| = \|\partial_\theta u\|_{L^\infty} \leq \frac{1}{T_* - t}.$$

- (v) The solution u converges asymptotically in the self-similar space to a self-similar solution to the Burgers equation Ψ_ν , given by (B.6). I.e.

$$\limsup_{s \rightarrow \infty} \|W - \Psi_\nu\|_{L^\infty} = 0.$$

- (vi) The solution has Hölder regularity $1/3$ at the blowup time, i.e. $u(x, T_*) \in C^{1/3}(\mathbf{R})$.

An example of an initial data satisfying these conditions and leading to a shock in finite time is $\varepsilon^{1/2}\eta(x)\Psi(x\varepsilon^{-3/2})$ for a suitable compactly supported smooth function η .

The gradient blowup outlined in Theorem 2.1 is stable and we can relax the initial conditions above and still satisfy the conclusions of Theorem 2.1:

Corollary 2.1.1. *There exists an open set of data in the H^6 topology such that the conclusions of Theorem 2.1 still hold.*

We prove Theorem (2.1) in section 5.3 and the corollary 2.1.1 in section 5.4.

3 Bootstraps: Assumptions and Consequences

Throughout we use \lesssim to denote a quantity which is bounded by a constant depending on α or h , but not on s, ε , or M . We establish our bootstrap assumptions in 3.1, derive consequences from these assumptions in 3.2, close our top order energy estimate in 3.3, and establish control on the Lagrangian trajectories in 3.4. In Section 4 we close our bootstraps in L^∞ .

3.1 Bootstrap Assumptions

Recall that $\widetilde{W} = W - \Psi$, where Φ is the stable Burgers profile (B.4). We assume the bootstraps

$$|\widetilde{W}|(y, s) \leq \begin{cases} \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)}\right) h^4, & 0 \leq |y| \leq h \\ \varepsilon^q \langle y \rangle^{1/3}, & h \leq |y| < \infty \end{cases} \quad (3.1a)$$

$$|\partial_y \widetilde{W}|(y, s) \leq \begin{cases} \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)}\right) h^3, & 0 \leq |y| \leq h \\ \varepsilon^{\frac{3}{4}\ell} \langle y \rangle^{-2/3}, & h \leq |y| \leq e^{ms} \\ 2e^{-s}, & e^{ms} \leq |y| \end{cases} \quad (3.2a)$$

$$\varepsilon^{\frac{3}{4}\ell} \langle y \rangle^{-2/3}, \quad h \leq |y| \leq e^{ms} \quad (3.2b)$$

$$2e^{-s}, \quad e^{ms} \leq |y| \quad (3.2c)$$

on the solution and the first derivative. For the higher derivatives we assume the following bootstraps on the Taylor region $0 \leq |y| \leq h$

$$|\partial_y^2 \widetilde{W}|(y, s) \leq \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)}\right) h^2, \quad (3.3a)$$

$$|\partial_y^3 \widetilde{W}|(y, s) \leq \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)}\right) h, \quad (3.3b)$$

$$|\partial_y^4 \widetilde{W}|(y, s) \leq \varepsilon^{\frac{7}{9}(1-3\alpha)}, \quad (3.3c)$$

for h given by (2.9) and m, ℓ, q given by (2.8).

To compensate for a loss of derivatives, we assume the following L^∞ control over the third derivative

$$\|\partial_y^3 W\|_{L^\infty} \leq M. \quad (3.4)$$

Finally, we assume the precise bootstrap

$$|\partial_y^3 W(0, s) - 6| \leq 10\varepsilon^{\frac{16}{9}(1-3\alpha)} < 1 \quad (3.5)$$

which will ensure that our solution converges to the correct profile. For the modulation variables we assume

$$|\tau(t)| \leq \varepsilon^{\frac{8}{9}(1-3\alpha)}, \quad |\xi(t)| \leq 4M\varepsilon, \quad (3.6a)$$

$$|\dot{\tau}(t)| \leq \varepsilon^{\frac{8}{9}(1-3\alpha)}, \quad |\dot{\xi}(t)| \leq 5M \quad (3.6b)$$

for all $t < T_*$.

3.2 Consequences of Bootstraps

The following are immediate consequences of the bootstraps (3.1)-(3.6).

1. The explicit property (B.8) of Ψ' along with (3.2) shows that $\partial_y W \in L_{y,s}^\infty(\mathbf{R})$ with the following uniform bound

$$\|\partial_y W\|_{L_{y,s}^\infty} = 2. \quad (3.7)$$

2. From the bootstrap (3.6b) on $\dot{\tau}$ we have the following bound on β_τ

$$\beta_\tau = \frac{1}{1 - \dot{\tau}} \leq \frac{1}{1 - \varepsilon^{\frac{1}{2}(1-3\alpha)}} \leq 1 + 2\varepsilon^{\frac{1}{2}(1-3\alpha)} \leq \frac{3}{2}. \quad (3.8)$$

This bound is true for $0 \leq \varepsilon < \frac{1}{8}$, and we will end up taking ε much smaller than this.

3. We have the following bound on κ :

$$|\kappa(t)| = |u(\xi(t), t)| \leq \|u\|_{L_{x,t}^\infty} \leq M. \quad (3.9)$$

This is shown with the following argument. Equation (1.1) satisfies a trivial maximum principle: if $\|u_0\|_{L^\infty} \leq M$ then $\|u\|_{L_{\theta,t}^\infty} \leq M$. This is an easy consequence of $(-\Delta)^\alpha u(x_0, t_0)$ being positive definite at a maximum. Then observe that $\kappa(t) = u(\xi(t), t)$ by (2.3) and the constraint (2.6).

4. The transformation (2.3) and the bound (3.9) on κ give

$$\|W\|_{L_y^\infty} \leq e^{s/2} (\|u_0\|_{L^\infty} + M). \quad (3.10)$$

The bootstrap (3.2) is insufficient to guarantee that $\partial_y W \in L^2$; we must use the structure of the equation (2.4).

Lemma 3.1. (Uniform L^2 Bound for $\partial_y W$). *The bootstraps (3.1)-(3.2c) imply that $\partial_y W$ is uniformly $L^2(\mathbf{R})$ in self-similar time.*

Proof. $\partial_y W$ solves (C.1), which we recall for convenience

$$\left(\partial_s + 1 + \beta_\tau \partial_y W \right) \partial_y W + g_W \partial_y W \partial_y^2 W = -\beta_\tau e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y W.$$

Taking the L^2 inner product with (C.1) gives

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \|\partial_y W\|_{L^2}^2 + \|\partial_y W\|_{L^2}^2 + \int_{\mathbf{R}} g_W \partial_y W \partial_y^2 W dy + \beta_\tau \int_{\mathbf{R}} (\partial_y W)^3 dy = \\ -\beta_\tau e^{(3\alpha-1)s} ((-\Delta)^\alpha \partial_y W, \partial_y W)_{L^2}. \end{aligned} \quad (3.11)$$

$(-\Delta)^\alpha$ is self-adjoint on L^2 and thus we have $((-\Delta)^\alpha \partial_y W, \partial_y W)_{L^2} \geq 0$. The transport term satisfies

$$\begin{aligned} \int_{\mathbf{R}} g_w \partial_y W \partial_y^2 W \, dy &= -\frac{1}{2} \int_{\mathbf{R}} g_w \frac{d}{dy} (\partial_y W)^2 \, dy \\ &= -\frac{\beta_\tau}{2} \int_{\mathbf{R}} (\partial_y W)^3 \, dy - \frac{3}{4} \|\partial_y W\|_{L^2}^2. \end{aligned} \quad (3.12)$$

From the uniform L^∞ bound (3.7) and the bootstraps (3.2b), (3.2c) it follows that

$$\begin{aligned} \int_{\mathbf{R}} (\partial_y W)^3 \, dy &= \left(\int_{0 \leq |y| \leq h} + \int_{h \leq |y| \leq e^{ms}} + \int_{e^{ms} \leq |y| < \infty} \right) (\partial_y W)^3 \, dy \\ &\leq C + \varepsilon^{\frac{3}{4}\ell} \int_{h \leq |y| \leq e^{ms}} \frac{1}{1+y^2} \, dy + 2 \int_{e^{ms} \leq |y| < \infty} e^{-s} |\partial_y W|^2 \, dy \\ &\leq C + 2 \arctan(e^{ms}) + 2e^{-s} \|\partial_y^2 W\|_{L^2}^2 \\ &\leq C + 2\varepsilon \|\partial_y W\|_{L^2}^2 \end{aligned} \quad (3.13)$$

where C is a constant which changes from line to line.

Combining (3.11) with (3.12)-(3.13) gives us

$$\begin{aligned} \frac{d}{ds} \|\partial_x W\|_{L^2}^2 &= \frac{C\beta_\tau}{2} + \left(\beta_\tau 2\varepsilon^b - \frac{1}{4} \right) \|\partial_x W\|_{L^2}^2 \\ &\leq \frac{C\beta_\tau}{2} - \frac{1}{8} \|\partial_x W\|_{L^2}^2 \end{aligned}$$

where the last inequality is obtained by taking ε sufficiently small and estimating β_τ using (3.8). Use Grönwall's inequality with the initial data assumption (2.15) to conclude. \square

3.3 Closure of Top Order Energy Estimate

The equation (1.1) suffers an L^∞ loss of derivatives; we cannot estimate $\partial_x^k u$ in L^∞ without control of $\partial_x^{k+1} u$ in L^∞ . To overcome this difficulty we use our bootstraps to linearize the evolution of the sixth derivative in L^2 and close the bootstrap (3.4) on the third derivative by interpolation (see Lemma A.2).

The evolution of $\partial_y^6 W$ is given by (C.4) which we reproduce for convenience

$$\left(\partial_s + \frac{11}{2} + 7\beta_\tau \partial_y W \right) \partial_y^6 W + g_w \partial_y^7 W = -\beta_\tau \left[e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^6 W + 35 \partial_y^3 W \partial_y^4 W + 21 \partial_y^2 W \partial_y^5 W \right].$$

We test in L^2 against $\partial_y^6 W$

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \|\partial_y^6 W\|_{L^2}^2 + \frac{17}{2} \|\partial_y^6 W\|_{L^2}^2 &= - \underbrace{\int_{\mathbf{R}} g_w \partial_y^7 W \partial_y^6 W \, dy}_{I} - 7\beta_\tau \int_{\mathbf{R}} \partial_y W (\partial_y^6 W)^2 \, dy \\ &\quad - 35\beta_\tau \int_{\mathbf{R}} \partial_y^3 W \partial_y^4 W \partial_y^6 W \, dy - 21\beta_\tau \underbrace{\int_{\mathbf{R}} \partial_y^2 W \partial_y^5 W \partial_y^6 W \, dy}_{II} \\ &\quad - \beta_\tau e^{(3\alpha-1)s} \underbrace{\int_{\mathbf{R}} \partial_y^6 W (-\Delta)^\alpha \partial_y^6 W \, dy}_{III}. \end{aligned}$$

The operator $(-\Delta)^\alpha$ is self-adjoint on L^2 , allowing us to drop III from subsequent estimates. Integration by parts and the supposed L^2 decay of $\partial_y^6 W$ gives

$$\begin{aligned} I &= \frac{1}{2} \int_{\mathbf{R}} g_w \frac{\partial}{\partial y} (\partial_y^6 W)^2 \, dy \\ &= \frac{\beta_\tau}{2} \left(\int_{\mathbf{R}} W \frac{\partial}{\partial y} (\partial_y^6 W)^2 \, dy + e^{s/2} (\kappa - \xi) \int_{\mathbf{R}} \frac{\partial}{\partial y} (\partial_y^6 W)^2 \, dy \right) + \frac{3}{4} \int_{\mathbf{R}} y \frac{\partial}{\partial y} (\partial_y^6 W)^2 \, dy \\ &= -\frac{\beta_\tau}{2} \int_{\mathbf{R}} \partial_y W (\partial_y^6 W)^2 \, dy - \frac{3}{4} \|\partial_y^6 W\|_{L^2}^2. \end{aligned}$$

Similarly, we form a total derivative and integrate by parts to obtain

$$II = \frac{21}{2} \beta_\tau \int_{\mathbf{R}} \partial_y^3 W (\partial_y^5 W)^2 dy.$$

Collecting like terms and using the bootstrap assumption (3.4) to bound the third derivative in L^∞ , we obtain the energy equality

$$\begin{aligned} \frac{d}{ds} \|\partial_y^6 W\|_{L^2}^2 &= -\frac{31}{2} \|\partial_y^6 W\|_{L^2}^2 - 13\beta_\tau \int_{\mathbf{R}} \partial_y W (\partial_y^6 W)^2 dy \\ &\quad - 70\beta_\tau \int_{\mathbf{R}} \partial_y^3 W \partial_y^5 W \partial_y^6 W dy + 21\beta_\tau \int_{\mathbf{R}} \partial_y^3 W (\partial_y^5 W)^2 dy - 2\|\partial_y^6 W\|_{H^{\alpha/2}} \\ &\leq \left(-\frac{31}{2} + 13\beta_\tau \right) \|\partial_y^6 W\|_{L^2}^2 + 35\beta_\tau M \|\partial_y^4 W\|_{L^2} \|\partial_y^6 W\|_{L^2} + 21\beta_\tau M \|\partial_y^5 W\|_{L^2}^2. \end{aligned} \quad (3.14)$$

We use Gagliardo-Nirenberg to interpolate between $\partial_y^6 W$ in L^2 and the uniform L^2 bound of $\partial_y W$ proven in Lemma 3.1:

$$\begin{aligned} \|\partial_y^4 W\|_{L^2} &\lesssim \|\partial_y W\|_{L^2}^{2/5} \|\partial_y^6 W\|_{L^2}^{3/5} \lesssim \|\partial_y W\|_{L^2}^{3/5}, \\ \|\partial_y^5 W\|_{L^2} &\lesssim \|\partial_y W\|_{L^2}^{1/5} \|\partial_y^6 W\|_{L^2}^{4/5} \lesssim \|\partial_y W\|_{L^2}^{4/5}. \end{aligned}$$

We insert the interpolants into our energy estimate (3.14), take β_τ close to 1 via (3.8), and apply Young's inequality to find

$$\frac{d}{ds} \|\partial_y^6 W\|_{L^2}^2 \lesssim \left(-\frac{31}{2} + 13\beta_\tau \right) \|\partial_y^6 W\|_{L^2}^2 + M\beta_\tau \|\partial_y^6 W\|_{L^2}^{8/5} \lesssim M - 2\|\partial_y^6 W\|_{L^2}^2.$$

Integrating this differential inequality gives

$$\|\partial_y^6 W\|_{L^2}^2 \lesssim \left(1 - e^{-2c(s-s_0)} \right) M + \|\partial_y^6 W^0\|_{L^2}^2 e^{-2c(s-s_0)}. \quad (3.15)$$

Using our initial data assumption (2.15) and taking M sufficiently large, we obtain

$$\|\partial_y^6 W\|_{L^2}^2 \lesssim M \leq M^{3/2}. \quad (3.16)$$

Finally, by Gagliardo-Nirenberg

$$\|\partial_y^3 W\|_{L^\infty} \lesssim \|\partial_y W\|_{L^\infty}^{5/9} \|\partial_y^6 W\|_{L^2}^{4/9} \lesssim M^{1/3}. \quad (3.17)$$

This closes the bootstrap (3.4) upon choosing M sufficiently large.

L^∞ Interpolation of the Intermediary Derivatives. We avoid bootstrapping the global bounds on $\partial_y^2 W$ and $\partial_y^3 W$ by interpolation. Gagliardo-Nirenberg gives the following bounds

$$\|\partial_y^5 W\|_{L^\infty} \lesssim \|\partial_y W\|_{L^\infty}^{1/9} \|\partial_y^6 W\|_{L^2}^{8/9} \lesssim M^{2/3} \quad (3.18a)$$

$$\|\partial_y^4 W\|_{L^\infty} \lesssim \|\partial_y W\|_{L^\infty}^{1/3} \|\partial_y^6 W\|_{L^2}^{2/3} \lesssim M^{1/2} \quad (3.18b)$$

$$\|\partial_y^2 W\|_{L^\infty} \lesssim \|\partial_y W\|_{L^\infty}^{7/9} \|\partial_y^6 W\|_{L^2}^{2/9} \lesssim M^{1/6} \quad (3.18c)$$

With control on $\partial_y^2 W$ and $\partial_y^3 W$ in L^∞ from (3.18c) and (3.17), we can combine the identity (2.7b) and interpolation inequality (A.3) for $(-\Delta)^\alpha$ to obtain

$$e^{s/2} |\kappa - \dot{\zeta}| \lesssim \frac{e^{(3\alpha-1)s}}{5} \|\partial_x^2 W\|_{L^\infty}^{1-2\alpha} \|\partial_x^3 W\|_{L^\infty}^{2\alpha} \leq e^{\frac{9}{10}(3\alpha-1)s}, \quad (3.19)$$

sacrificing powers of ε to absorb the constant.

3.4 Lagrangian Trajectories

We prove upper and lower bounds on the Lagrangian trajectories of (2.4) which allow us to convert spatial decay into temporal decay.

The flowmap evolution of (2.4) is

$$\frac{d}{ds}\Phi^{y_0} = \frac{3}{2}\Phi^{y_0} + \beta_\tau \left(W^{y_0} + e^{\frac{\kappa}{2}}(\kappa - \dot{\xi}) \right), \quad (3.20)$$

where Φ^{y_0} is the trajectory emanating from the label y_0 and $W^{y_0}(s) := W(\Phi^{y_0}(s), s)$.

Lemma 3.2. (Upper Bound on Trajectories). *Let $x_0 \in \mathbf{R}$, then trajectories are bounded above by*

$$|\Phi^{x_0}(s)| \leq \left(|x_0| + \frac{3}{2}C\varepsilon^{-1/2} \right) e^{\frac{3}{2}(s-s_0)}, \quad (3.21)$$

with C a positive constant depending linearly on M .

Proof. The flowmap equation (3.20) implies that

$$\frac{d}{ds} \left[\Phi^{x_0} e^{-3s/2} \right] = \beta_\tau e^{-3s/2} \left(W^{x_0} + e^{s/2}(\kappa - \dot{\xi}) \right).$$

We integrate in time, estimating β_τ with (3.8), and apply (3.10), (3.19) to obtain

$$\begin{aligned} |\Phi^{x_0}(s)| &\leq |x_0| e^{\frac{3}{2}(s-s_0)} + e^{3s/2} \int_{s_0}^s e^{-\frac{3}{2}s'} \beta_\tau \left(W^{x_0}(s') + e^{s'/2}(\kappa - \dot{\xi}) \right) ds' \\ &\leq \varepsilon |x_0| e^{\frac{3}{2}s} + \frac{3}{2} \varepsilon^{\frac{3}{2}s} \int_{s_0}^s e^{-s'} (\|u_0\|_{L^\infty} + 6M) + e^{-\frac{3}{2}s'} \varepsilon^{\frac{9}{10}(1-3\alpha)} ds', \end{aligned}$$

from which the result follows. \square

Next we prove two lower bounds on the Lagrangian trajectories. The first bound serves only to guarantee that all trajectories emanating from outside the Taylor region will eventually take off with our optimal bound (3.23).

Lemma 3.3. *Let $h \leq |x_0| < \infty$. The trajectories are bounded below by*

$$|\Phi^{x_0}(s)| \geq |x_0| e^{\frac{1}{5}(s-s_0)}. \quad (3.22)$$

Proof. The proof is contained in [Yan20], replacing the author's bound for $e^{\frac{\kappa}{2}}(\kappa - \dot{\xi})$ by our bound (3.19) and taking ε sufficiently small. \square

To close the bootstraps, this $1/5$ bound proves insufficient; we require a stronger lower bound to prove Lemma 4.1 and to close the bootstrap (3.1b). In particular we must have that the trajectories eventually take off faster than $e^{\frac{3}{5}s}$. The following estimate is optimal in light of Lemma 3.2 and the transformation (2.2).

Lemma 3.4. *Let $1 \leq |x_0| < \infty$. The trajectories are bounded below by*

$$|\Phi^{x_0}(s)| \geq \left(|x_0|^{2/3} - \frac{2C}{3} \right) e^{\frac{3}{2}(s-s_0)} > 0. \quad (3.23)$$

The constant C is

$$C = (1 + 2\varepsilon^{\frac{1}{2}(1-3\alpha)}) \left(1 + \frac{1}{h^{1/3}} \left(\varepsilon^q \langle h \rangle + \varepsilon^{\frac{1}{2}(1-3\alpha)} \right) \right).$$

Proof. Multiply the flowmap (3.20) by Φ^{x_0} to obtain

$$\frac{d}{ds} |\Phi^{x_0}|^2 = 3|\Phi^{x_0}|^2 + 2\beta_\tau \Phi \left(W + e^{\frac{\kappa}{2}}(\kappa - \dot{\xi}) \right).$$

Recall that Ψ is the self-similar solution to Burgers equation (B.1) and observe that in light of (B.7) and (3.1b) we have

$$|W(x, s)| \leq |\Psi(x)| + |\widetilde{W}(x, s)| \leq |x|^{1/3} + \varepsilon^q \langle x \rangle^{1/3} \leq \left(1 + \varepsilon^q \frac{\langle h \rangle^{1/3}}{h^{1/3}} \right) |x|^{1/3}.$$

In a similar vein, (3.19) implies that

$$|e^{\frac{s}{2}}(\kappa - \dot{\xi})| \leq \frac{\varepsilon^{\frac{1}{2}(1-3\alpha)}}{h^{1/3}} |x^{1/3}|.$$

Estimating β_τ as in (3.8) and taking C as in the statement of the lemma, we obtain the differential inequality

$$\frac{d}{ds} |\Phi^{x_0}|^2 - 3|\Phi^{x_0}|^2 \geq -2C|\Phi^{x_0}|^{4/3}.$$

This inequality is separable in the variable $|\Phi^{x_0}|^2 e^{-3s}$, and integrating from s_* , $s_0 \leq s_* < s$ yields (3.23) upon taking ε sufficiently small. \square

Together, these two lemmas show that all trajectories with starting labels $h \leq |x_0|$ will eventually take off like $e^{\frac{3}{2}s}$, i.e. $|\Phi^{x_0}(s)| \lesssim e^{\frac{3}{2}s}$ whenever $h \leq |x_0|$.

4 Bootstraps: Closing The L^∞ Estimates

4.1 Modulation Variables

Our interpolation inequality (A.3) for $(-\Delta)^\alpha$ and the estimates (3.7), (3.18c) give

$$|\dot{\tau}| \leq e^{(3\alpha-1)s} \|\partial_y W\|_{L^\infty}^{1-2\alpha} \|\partial_y^2 W\|_{L^\infty}^{2\alpha} \lesssim M^{\frac{1}{3}\alpha} e^{(3\alpha-1)s} \leq \frac{1}{2} e^{\frac{8}{9}(3\alpha-1)s} \quad (4.1)$$

upon choosing ε sufficiently small. We apply the fundamental Theorem of calculus and recall that $\tau(-\varepsilon) = 0$ to find

$$|\tau(t)| \leq \int_{-\varepsilon}^t |\dot{\tau}(t')| dt' \leq \frac{1}{2} \varepsilon^{\frac{8}{9}(1-3\alpha)} (T_* + \varepsilon) \leq \frac{1}{2} \varepsilon^{1-3\alpha} + \frac{1}{2} \varepsilon^{1+\frac{1}{2}(1-3\alpha)} \leq \frac{3}{4} \varepsilon^{\frac{8}{9}(1-3\alpha)}.$$

This closes the τ bootstraps (3.6a) and (3.6b).

The ξ bootstrap is equally simple. By the identity (2.7b),

$$\begin{aligned} |\dot{\xi}| &\leq |\kappa| + \frac{e^{s(3\alpha-\frac{3}{2})}}{|\partial_y^3 W(0, s)|} |((-\Delta)^\alpha \partial_y^2 W)(0, s)| \\ &\leq M + \frac{1}{5} e^{(3\alpha-\frac{3}{2})s} M^{\frac{1}{6}+\frac{\alpha}{3}} \\ &\leq 2M, \end{aligned} \quad (4.2)$$

where we have chosen ε sufficiently small and M sufficiently large. Integrating (4.2) in time and using the fundamental theorem of calculus, recalling that $|\xi(-\varepsilon)| = 0$,

$$|\xi(t)| \leq \int_{-\varepsilon}^t |\dot{\xi}| dt' \leq 2M(T_* + \varepsilon) \leq 2M(\varepsilon^{1+2\alpha} + \varepsilon) \leq 4M\varepsilon.$$

This closes the two ξ bootstraps (3.6a) and (3.6b).

4.2 Taylor Region

(Fourth Derivative on $0 \leq |y| \leq h$). The fourth derivative of \widetilde{W} evolves according to (C.7), which we recall here

$$(\partial_s + \underbrace{\frac{11}{2} + 5\beta_\tau \partial_y W}_{\mathcal{D}(y)}) \partial_y^4 \widetilde{W} + g_W \partial_y^4 \widetilde{W} = -\beta_\tau \mathcal{F}.$$

\mathcal{D} is the damping and \mathcal{F} is the forcing, which is given explicitly in (4.4).

Using (3.7) and (3.8) we bound the damping below by

$$\mathcal{D} \geq \frac{11}{2} - 5\beta_\tau \geq \frac{1}{4}. \quad (4.3)$$

The forcing is given by

$$\begin{aligned} \mathcal{F} := & \underbrace{-\beta_\tau e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^4 W}_I + \underbrace{\beta_\tau \left(\widetilde{W} \Psi^{(5)} + 4\partial_y \widetilde{W} \Psi^{(4)} + 8\partial_y^2 \widetilde{W} \Psi^{(3)} + 10\partial_y^3 \widetilde{W} \Psi'' + 11\partial_y^2 \widetilde{W} \partial_y^3 \widetilde{W} \right)}_{II} \\ & + \underbrace{\beta_\tau e^{s/2} (\kappa - \dot{\xi}) \partial_y^5 \Psi}_{III} + \underbrace{\beta_\tau \dot{\tau} \left(\Psi \partial_y^5 \Psi + 5\Psi' \Psi^{(4)} + 10\Psi'' \Psi^{(3)} \right)}_{IV} \end{aligned} \quad (4.4)$$

Term I is estimated by combining our interpolation estimate (A.3) on the fractional Laplacian with Gagliardo-Nirenberg

$$|I| \lesssim \beta_\tau e^{(3\alpha-1)s} \|\partial_y W\|_{L^\infty}^{\frac{1}{3}-\frac{4}{9}\alpha} \|\partial_y^6 W\|_{L^2}^{\frac{2}{3}+\frac{4}{9}\alpha} \leq \beta_\tau e^{(3\alpha-1)s} M^{\frac{1}{2}+\frac{\alpha}{3}} \leq \varepsilon^{\frac{8}{9}(1-3\alpha)},$$

where in the second inequality we have used (3.7) and (3.16) and in the final inequality we have sacrificed a power of ε . Recalling that $h = (\log M)^{-2}$, the term II can be bounded by the bootstrap assumptions (3.1)-(3.3) in the Taylor region

$$\begin{aligned} |II| & \lesssim \varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)} (h^4 + h^3 + h^2 + h) \\ & \lesssim \varepsilon^{\frac{8}{9}(1-3\alpha)} + (\log M)^{-1} \varepsilon^{\frac{7}{9}(1-3\alpha)}. \end{aligned}$$

The estimate (3.19) along with the explicit property (B.8) on $\Psi^{(5)}$ imply that

$$|III| = \left| e^{s/2} (\kappa - \dot{\xi}) \Psi^{(5)} \right| \lesssim \varepsilon^{\frac{9}{10}(1-3\alpha)} \leq \varepsilon^{\frac{8}{9}(1-3\alpha)}.$$

The term IV is quadratic in Ψ so (3.6) and (B.8) give

$$|IV| \lesssim \varepsilon^{\frac{1}{2}(1-3\alpha)} \leq \varepsilon^{\frac{8}{9}(1-3\alpha)}.$$

In light of (4.3) and our forcing estimates, we write

$$\partial_s \partial_y^4 \widetilde{W} + \frac{1}{4} \partial_y^4 \widetilde{W} + g_W \partial_y^4 \widetilde{W} \geq -\beta_\tau \mathcal{F}.$$

Composing this inequality with the flowmap Φ^{y_0} and integrating gives us

$$\begin{aligned} |\partial_y^4 \widetilde{W}^{y_0}(s)| & \leq |\partial_y^4 \widetilde{W}^{y_0}(s)| e^{-\frac{s}{4}} + e^{-\frac{s}{4}} \int_{s_0}^s e^{\frac{s'}{4}} \mathcal{F}^{y_0} ds' \\ & \leq \varepsilon^{1-3\alpha} + C(\varepsilon^{\frac{8}{9}(1-3\alpha)} + (\log M)^{-1} \varepsilon^{\frac{7}{9}(1-3\alpha)}) \\ & \leq \frac{1}{2} \varepsilon^{\frac{7}{9}}, \end{aligned} \quad (4.5)$$

where we have used the Lagrangian trajectory lower bound (3.23) and the initial condition (2.14). This closes the bootstrap (3.3c) for $\partial_y^4 \widetilde{W}$.

(Third Derivative at 0). The third derivative evolves according to equation (C.3), which reads

$$\left(\partial_s + 4 + 4\beta_\tau \partial_y W \right) \partial_y^3 W + g_W \partial_y^4 W = -\beta_\tau \left[e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^3 W + 3(\partial_y^2 W)^2 \right].$$

Setting $y = 0$ and using the constraints (2.6) we obtain the identity

$$\partial_s \partial_y^3 W^0 = \underbrace{4\dot{\tau} \beta_\tau \partial_y^3 W^0}_I - \underbrace{\beta_\tau e^{s/2} (\kappa - \dot{\xi}) \partial_y^4 W^0}_{II} - \underbrace{\beta_\tau e^{(3\alpha-1)s} (-\Delta)^\alpha [\partial_y^3 W]^0}_{III}.$$

Using our β_τ estimate (3.8), the fact that $|\partial_y^3 W^0(s)| \leq 7$, and the $\dot{\tau}$ bootstrap closure (4.1), we find

$$|I| \leq 28e^{\frac{8}{9}(3\alpha-1)s}.$$

Next we apply the estimates (3.19) and (3.8), as well as our L^∞ control (3.18b) on $\partial_y^4 W$ to find

$$|II| \lesssim 2M^{\frac{1}{2}} e^{\frac{9}{10}(3\alpha-1)s} \leq e^{\frac{8}{9}(3\alpha-1)s}$$

upon taking ε sufficiently small. Finally, we estimate

$$|III| \lesssim 2M^{\frac{1}{3}(1+\alpha)} (1-\alpha) e^{(3\alpha-1)s} \leq e^{\frac{8}{9}(3\alpha-1)s}$$

by applying our interpolation estimate (A.3) for the fractional Laplacian, (3.8), our L^∞ bounds (3.17), (3.18b), and taking ε sufficiently small.

We record the following fact

$$|\partial_s \partial_y^3 W(0, s)| \leq 3e^{\frac{8}{9}(3\alpha-1)s} \leq 3\varepsilon^{\frac{8}{9}(1-3\alpha)}, \quad (4.6)$$

which we will need to use later in our proof.

Recalling that $\partial_y^3 W(0, s_0) = 6$ and applying the Fundamental theorem of calculus to the estimate above, we see that

$$\begin{aligned} |\partial_y^3 W(0, s) - 6| &\leq \int_{-\varepsilon}^t 3\varepsilon^{\frac{8}{9}(1-3\alpha)} dt' \leq 3\varepsilon^{\frac{8}{9}(1-3\alpha)} (T_* + \varepsilon) \leq 3\varepsilon^{(\frac{8}{9} + \frac{8}{9})(1-3\alpha)} + 3\varepsilon^{(1 + \frac{8}{9})(1-3\alpha)} \\ &\leq 6\varepsilon^{\frac{16}{9}(1-3\alpha)}. \end{aligned} \quad (4.7)$$

Taking ε small closes the bootstrap (3.5).

(Zeroth, First, Second, and Third Derivative on $0 \leq |y| \leq h$). By the fundamental theorem of calculus, our bootstrap (3.5), and the estimate (4.5),

$$\begin{aligned} |\partial_y^3 \widetilde{W}| &= \left| \partial_y^3 \widetilde{W}(0, s) + \int_0^y \partial_y^4 \widetilde{W}(y', s) dy' \right| \\ &\leq 6\varepsilon^{\frac{9}{10}(1-3\alpha)} + \frac{1}{2} \varepsilon^{\frac{7}{9}(1-3\alpha)} h \\ &\leq \frac{1}{2} \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)} \right) h, \end{aligned}$$

which closes (3.3b) on $\partial_y^3 \widetilde{W}$. Recall that our constraints (2.6) at the origin are

$$\partial_y^j \widetilde{W}(0, s) = 0$$

for $j = 0, 1, 2$. We can iterative apply the fundamental theorem of calculus to estimate

$$|\partial_y^j \widetilde{W}| = \left| \partial_y^j \widetilde{W}(0, s) + \int_0^y \partial_y^{j+1} \widetilde{W}(y', s) dy' \right| \leq \frac{1}{2} \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)} \right) h^{4-j},$$

which closes the bootstraps (3.1a), (3.2a), and (3.3a).

4.3 Outside the Taylor Region

To close the bootstraps on the rest of \mathbf{R} we will utilize the trajectory framework presented in Appendix A.1.

(Zeroth Derivative on $h \leq |y| < \infty$). We consider the weighted variable $V := \langle x \rangle^{-1/3} \widetilde{W}$. The evolution equation for V is obtained from (C.5) after a short computation

$$\left(\partial_s - \underbrace{\frac{1}{2} + \beta_\tau \Psi' + \frac{g_W}{3} y \langle y \rangle^{-2}}_{\mathcal{D}(y)} \right) V + g_W V' = -\beta_\tau \langle y \rangle^{-\frac{1}{3}} F_{\widetilde{W}}, \quad (4.8)$$

where the forcing is given below in (4.10). Closing the bootstrap (3.1b) now amounts to showing that $|V(x, s)| < \varepsilon^q$ uniformly in self similar time, for $h \leq |x| < \infty$. From Appendix A.1, if we can control the damping from below and the forcing from above on our region $h \leq |x| < \infty$, then we can establish global (in time) estimates by carefully following the trajectories through our region.

We begin by bounding our damping below by (3.8), (B.8), (3.19) and our bootstrap assumption (3.1b),

$$\begin{aligned}
\mathcal{D}(y) &\geq \frac{1}{2} - (1 + 2\varepsilon^{\frac{1}{2}(1-3\alpha)})\langle y \rangle^{-2/3} - \frac{1}{3} \left(\beta_\tau W + \beta_\tau e^{s/2}(\kappa - \dot{\xi}) + \frac{3}{2}y \right) y \langle y \rangle^{-2} \\
&\geq \frac{1}{2} \langle y \rangle^{-2} - (1 + 2\varepsilon^{\frac{1}{2}(1-3\alpha)})\langle y \rangle^{-2/3} - \frac{2}{3}(1 + \varepsilon^q)y \langle y \rangle^{-5/3} - \varepsilon^{\frac{1}{2}(1-3\alpha)}|y| \langle y \rangle^{-2} \\
&\geq -(2 + 3\varepsilon^{\frac{1}{2}(1-3\alpha)} + \frac{2}{3}\varepsilon^{\frac{2}{3}(1-3\alpha)})\langle y \rangle^{-2/3} \\
&\geq -3\langle y \rangle^{-2/3}.
\end{aligned}$$

This inequality holds upon taking ε sufficiently small.

Let s_* be the first time that the Trajectory Φ^{y_0} enters the region $h \leq |y| < \infty$. By composing the damping with the lower bound of the trajectories and integrating in time we find that

$$\int_{s_*}^s \langle \Phi^{y_0}(s') \rangle^{-\frac{2}{3}} ds' \leq \int_{s_*}^{\infty} \langle |y_0| e^{\frac{2}{3}(s-s_*)} \rangle^{-2/3} ds' \leq \frac{13}{2} \log\left(\frac{1}{h}\right), \quad (4.9)$$

which is the same estimate obtained in [BSV19a]. Using (4.9), we set

$$\lambda_{\mathcal{D}} := e^{-\int_{s_0}^s \mathcal{D}(\Phi^{y_0}(s')) ds'} \leq e^{\frac{39}{2} \log(\frac{1}{h})} = h^{-\frac{39}{2}}.$$

Next we bound the forcing above in L^∞ . Recall that the forcing for equation (4.8) is given by (C.5)

$$F_{\bar{W}} = \underbrace{e^{-\frac{s}{2}\dot{\kappa}}}_I + \underbrace{e^{(3\alpha-1)s}(-\Delta)^\alpha W}_{II} + \underbrace{\Psi'(\dot{\tau}\Psi)}_{III} + \underbrace{\Psi' e^{\frac{s}{2}}(\kappa - \dot{\xi})}_{IV}. \quad (4.10)$$

We also note that $\langle y \rangle^{-1/3} \circ \Phi^{y_0}(s) \leq e^{-\frac{1}{2}(s-s_0)}$ as a consequence of our lower bound (3.23) on the trajectories.

We estimate term by term. For the first term we use identity (2.7b), the interpolation inequality (A.3), (3.8), (3.10), and take ε sufficiently small

$$\begin{aligned}
\beta_\tau |\langle \Phi^{y_0}(s) \rangle^{-1/3} I| &\lesssim \beta_\tau \langle e^{\frac{3}{2}(s-s_*)} \rangle^{-1/3} e^{(3\alpha-1)s} \left[(-\Delta)^\alpha W^0 - \frac{(-\Delta)^\alpha \partial_y^2 W^0}{\partial_y^3 W^0} \right] \\
&\lesssim \beta_\tau \varepsilon^{\frac{1}{2}} e^{(3\alpha-\frac{3}{2})s} \left(\|W\|_{L^\infty}^{1-2\alpha} \|\partial_y W\|_{L^\infty}^{2\alpha} + \frac{1}{5} \|\partial_y^2 W\|_{L^\infty}^{1-2\alpha} \|\partial_y^3 W\|_{L^\infty}^{2\alpha} \right) \\
&\lesssim \varepsilon^{\frac{1}{2}} e^{(3\alpha-\frac{3}{2})s} \left(e^{\frac{s}{2}-\alpha s} + \frac{1}{5} M^{\frac{1}{6}-\frac{\alpha}{3}} \right) \\
&\leq \frac{1}{4} e^{(2\alpha-1)s}.
\end{aligned}$$

Note that the $e^{s/5}$ lower bound (3.22) on the trajectories is insufficient to close the above estimate.

For the next term we use our interpolation estimate (A.3), (3.10), (3.8), and sacrifice powers of ε to get

$$\beta_\tau |\langle \Phi^{y_0}(s) \rangle^{-1/3} II| \lesssim \varepsilon^{1/2} e^{(3\alpha-\frac{3}{2})s} \|W\|_{L^\infty}^{1-2\alpha} \|\partial_y W\|_{L^\infty}^{2\alpha} \leq \frac{1}{4} e^{(2\alpha-1)s}.$$

We estimate the third term by using (B.8) twice, (3.8), our bootstrap assumption (3.6) on $\dot{\tau}$, and taking ε small

$$\beta_\tau |\langle \Phi^{y_0}(s) \rangle^{-1/3} III| \lesssim \dot{\tau} \varepsilon^{1/2} e^{-\frac{s}{2}} \langle \Phi^{y_0}(s) \rangle^{-\frac{1}{3}} \leq \frac{1}{4} e^{-s}.$$

For the final term we use (3.19), (B.8), and take ε small to find

$$\beta_\tau |\langle \Phi^{y_0}(s) \rangle^{-1/3} IV| \leq 2\varepsilon^{1/2} \frac{e^{(3\alpha-\frac{3}{2})s}}{5} \|\partial_y^2 W\|_{L^\infty}^{1-2\alpha} \|\partial_y^3 W\|_{L^\infty}^{2\alpha} \langle \Phi^{y_0}(s) \rangle^{-2/3} \leq \frac{1}{4} e^{(3\alpha-\frac{5}{2})s}.$$

Together these four estimates give

$$|\beta_\tau \langle y \rangle^{-1/3} F_{\bar{W}}| \leq e^{(2\alpha-1)s}. \quad (4.11)$$

Since $\alpha < 1/3$, we have temporal decay in our forcing terms. Inserting the damping estimate (4.9) and the forcing estimate (4.11) into (A.2) gives

$$\begin{aligned} |V^{y_0}(s)| &\lesssim \lambda_D |V(y_0)| + \lambda_D \int_{s_*}^s e^{-(1-2\alpha)s'} ds' \\ &\lesssim \lambda_D \left(|V(y_0)| + e^{(2\alpha-1)s_*} \right) \end{aligned}$$

As in the discussion in Appendix A.1, we have two cases; either $s_* = s_0$, and $h \leq y_0 < \infty$, or $s_* > s_0$ and $y_0 = h$. In the first case we use our initial data assumption (2.12) to find that

$$|V^{y_0}(s)| \lesssim \lambda_D \left(\varepsilon^{1-3\alpha} \langle y_0 \rangle^0 + \varepsilon^{-(1-2\alpha)s_0} \right) \leq \varepsilon^{\frac{1}{2}(1-2\alpha)}.$$

In the second case we apply our bootstrap (3.1a) for the region $0 \leq |y| \leq h$ and estimate

$$|V^{y_0}(s)| \lesssim \lambda_D \left(\left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)} \right) h^4 + \varepsilon^{-(1-2\alpha)s_*} \right).$$

In both cases we may take ε sufficiently small to get

$$|V^{y_0}(s)| \leq \frac{1}{2} \varepsilon^q.$$

This closes the zeroth derivative bootstrap (3.1) over the region $h \leq |y| < \infty$.

(First Derivative on $h \leq |y| \leq e^{ms}$). We use the same strategy used to close the zeroth derivative bootstrap. Making the weighted change of variables $V := \langle y \rangle^{2/3} \partial_y \widetilde{W}$, a short computation using (C.6) yields the following evolution equation for V

$$\partial_s V + \underbrace{\left(1 + \beta_\tau (\partial_y \widetilde{W} + 2\Psi') - \frac{2}{3} y \langle y \rangle^{-2} g_W \right)}_{\mathcal{D}(y)} V + g_W \partial_y V = -\beta_\tau \langle y \rangle^{2/3} F_{\partial_y \widetilde{W}} \quad (4.12)$$

where $F_{\widetilde{W}}$ is the forcing term from (C.6) which is given below in (4.14). Closing the bootstrap (3.2b) now amounts to showing that $|V(x, s)| < \varepsilon^{\frac{3}{4}\ell}$ uniformly in s for all $h \leq |x| \leq e^{ms}$.

We begin by bounding the damping below. Using (3.8), (B.8), the bootstraps (3.1) and (3.2), and (3.19) we find

$$\begin{aligned} \mathcal{D}(y) &= -1 - \beta_\tau (\partial_y \widetilde{W} + 2\Psi') + \frac{2}{3} y \langle y \rangle^{-2} \left(\frac{3}{2} y + \beta_\tau \left(W + e^{s/2} (\kappa - \dot{\xi}) \right) \right) \\ &\geq -\langle y \rangle^{-2} - (1 + 2\varepsilon^{\frac{1}{2}(1-3\alpha)}) \left(\varepsilon^{\frac{3}{4}\ell} \langle y \rangle^{-2/3} + 2\langle y \rangle^{-2/3} \right. \\ &\quad \left. + \frac{2}{3} |y| \langle y \rangle^{-5/3} + \varepsilon^{\frac{1}{2}(1-3\alpha)} \frac{2}{3} (1+)|y| \langle y \rangle^{-2} \right) \\ &\geq -\langle y \rangle^{-2} - 5\langle y \rangle^{-2/3} \\ &\geq -6\langle y \rangle^{-2/3}. \end{aligned}$$

Our estimate (4.9) still applies to the current damping lower bound for the first derivative. Therefore we have

$$\lambda_{\mathcal{D}} := e^{-\int_{s_0}^s \mathcal{D}(\Phi^{y_0}(s')) ds'} \leq e^{39 \log(\frac{1}{h})} = h^{-39}. \quad (4.13)$$

The forcing for V in this case is given by

$$F_{\partial_y \widetilde{W}} := \underbrace{e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y W}_I + \underbrace{e^{s/2} (\kappa - \dot{\xi}) \Psi''}_{II} + \underbrace{\widetilde{W} \Psi''}_{III} + \underbrace{\dot{\tau} \Psi \Psi''}_{IV} + \underbrace{(\Psi')^2}_V \quad (4.14)$$

We bound the first term using (3.8), the interpolation estimate (A.3), and by taking ε sufficiently small.

$$\begin{aligned} \beta_\tau |\langle y \rangle^{2/3} I| &\lesssim e^{\frac{2}{3}ms} e^{(3\alpha-1)s} \|\partial_y W\|_{L^\infty}^{1-2\alpha} \|\partial_y^2 W\|_{L^\infty}^{2\alpha} \\ &\lesssim M^{\frac{1}{6}} e^{\left(\frac{2}{3}m+3\alpha-1\right)s}. \end{aligned}$$

We choose m such that the exponent in the exponential above remains positive for all $0 < \alpha < 1/3$. Remember that we are aiming to prove that V is bounded by some multiple of $\varepsilon^{2/3(1-3\alpha)}$, which means we must have $m < \frac{3}{8}(1-3\alpha)$. This informs the choice of m in (2.8).

For the second term, we use identity (2.7b), (B.8), the interpolation estimate (A.3), the L^∞ control (3.17),(3.18b), and then take ε small to find

$$\begin{aligned} \beta_\tau |\langle y \rangle^{2/3} II| &\leq \beta_\tau \frac{e^{(3\alpha-1)s}}{\partial_y^3 \widetilde{W}(0, s)} |(-\Delta)^\alpha \partial_y^2 W(0, s)| \langle y \rangle^{-1} \\ &\lesssim M^{\frac{1}{6}(1+\alpha)} e^{(3\alpha-1-\frac{3}{2})s}. \end{aligned}$$

We apply the bootstrap (3.1a) for \widetilde{W} , use (B.8), and estimate the trajectories with (3.22) to get

$$\begin{aligned} \beta_\tau \left(\langle y \rangle^{2/3} III \right) \circ \Phi^{y_0}(s) &\leq \beta_\tau \varepsilon^{\frac{1}{2}q} \langle \Phi^{y_0}(s) \rangle^{-2/3} \\ &\lesssim \varepsilon^{\frac{1}{2}q} e^{-s}. \end{aligned}$$

Next, (B.8), (3.8), and taking ε small yields

$$\beta_\tau \left(\langle y \rangle^{2/3} IV \right) \circ \Phi^{y_0}(s) \leq \beta_\tau \dot{\tau} \langle \Phi^{y_0}(s) \rangle^{-4/3} \lesssim \varepsilon^{\frac{8}{3}(1-3\alpha)} e^{-2s}.$$

Finally, (B.8), (3.8), and (3.6b) imply that

$$\beta_\tau \left(\langle y \rangle^{2/3} V \right) \circ \Phi^{y_0}(s) \leq \beta_\tau \langle \Phi^{y_0}(s) \rangle^{-2/3} \lesssim e^{-s}.$$

To summarize, we have obtained the following forcing bound

$$|F_{\partial_y \widetilde{W}} \circ \Phi^{y_0}(s)| \lesssim e^{-\ell s}, \quad (4.15)$$

with ℓ as in (2.8).

We once again apply the trajectory framework from Appendix A.1 to our equation (4.12) for V . Plugging our damping estimate (4.13) and our forcing estimate (4.15) into (A.2) we obtain

$$\begin{aligned} |V^{y_0}(s)| &\lesssim h^{-39} |V(y_0)| + \lambda_D \int_{s_*}^s e^{-\ell s'} ds' \\ &\lesssim h^{-39} \left(|V(y_0)| + e^{-\ell s_*} \right) \end{aligned}$$

As in our closure of the zeroth derivative bootstrap and the discussion of our transport framework from Appendix A.1, we have two cases. In the first case, $s_* = s_0$ and $h \leq y_0 \leq \varepsilon^{-m}$ and we use our initial data assumption (2.13) to find that

$$|V^{y_0}(s)| \lesssim h^{-39} \left(\varepsilon^{\frac{3}{4}\ell} \langle y_0 \rangle^{-2/3} + \varepsilon^\ell \right) \lesssim M^{20} \left(\varepsilon^{\frac{3}{4}\ell} + \varepsilon^\ell \right) \leq \frac{3}{4} \varepsilon^{\frac{3}{4}\ell},$$

upon choosing ε sufficiently small.

In the second case we have that $s_* > s_0 = -\log \varepsilon$, and (3.2) means $V^{y_0}(s_*) = V(h)$. This gives

$$|V^{y_0}(s)| \lesssim M^{20} \left(\varepsilon^{\frac{2}{3}(1-3\alpha)} h^2 \langle h \rangle^{-2} + \varepsilon^\ell \right) \leq \frac{3}{4} \varepsilon^{\frac{3}{4}\ell}$$

(since $\varepsilon^{\frac{3}{4}\ell} < \varepsilon^{\frac{2}{3}(1-3\alpha)}$ for all α). Thus we have closed the first derivative bootstrap (3.2) on the region $h \leq |y| \leq e^{ms}$.

(First Derivative on $e^{ms} \leq |x| < \infty$). We prove a short fact about the temporal decay of the fractional Laplacian along trajectories.

Lemma 4.1. *Suppose $e^{ms} \leq |y| < \infty$. Then*

$$(-\Delta)^\alpha [\partial_y W](\Phi^y(s)) \lesssim e^{-s}$$

for all s sufficiently large.

Proof. We use the singular integral representation (1.2) for $(-\Delta)^\alpha$. Let $(-\Delta)^\alpha_{[a,b]} u := C_\alpha \int_{a \leq |\eta| \leq b} \cdots d\eta$ and decompose the fractional Laplacian as follows

$$(-\Delta)^\alpha \partial_y W = \underbrace{(-\Delta)^\alpha_{[0,h]} \partial_y \widetilde{W}}_I + \underbrace{(-\Delta)^\alpha_{[h,e^{ms}]} \partial_y \widetilde{W}}_{II} + \underbrace{(-\Delta)^\alpha_{[e^{ms},\infty)} \partial_y W}_{III} + \underbrace{(-\Delta)^\alpha_{[0,e^{ms}]} \Psi'}_{IV}.$$

The term IV is majorized by (B.9), that is $|IV| \leq \langle y \rangle^{-2/3-2\alpha}$. Composing with the lower bound (3.23) on the trajectories shows that $|IV| \lesssim e^{-(1+3\alpha)s}$.

We now go term by term and bound using our bootstraps (3.2). Using (3.2a) for the first term we have

$$\begin{aligned} |I| &\leq C_\alpha \int_{0 \leq |\eta| \leq h} \frac{e^{-s} + C}{|y - \eta|^{1+2\alpha}} d\eta \\ &\leq 2hC_\alpha (e^{-s} + C) |y - h|^{-1-2\alpha}, \end{aligned}$$

where C is the constant from (3.2a).

Trajectories will either satisfy $\Phi^{y_0}(s) > e^{ms}$ for $|y_0| \geq e^{ms_0}$, or s_* is the first time that a trajectory enters $e^{ms} \leq y < \infty$, in which case $|y_*| = e^{ms_*}$. In both cases since $e^{ms_0} > 1$ we can apply our trajectory lower bound (3.23). Since $y = \Phi^{y_0}(s) \geq |y_0| > e^{ms} > 1 > h$ we have that $|y - h| > |y|$ and composing with trajectories gives $|y - h| \gtrsim e^{\frac{3}{2}s}$. Therefore

$$|I| \lesssim e^{-\frac{3}{2}(1+2\alpha)s}.$$

We remark that this estimate required a lower bound on the trajectories of at least $e^{\frac{3}{5}s}$.

The integrand becomes singular at $y = e^{ms}$. Since $\partial_y W$ and $\partial_y^2 W$ are both L^∞ , the interpolation inequality (A.3) guarantees that the integral converges. Because we only care about the asymptotic behavior of these quantities, and since we will be composing with trajectories, we can safely assume that $|x| \geq e^{ms} + 1$. The same argument as before applies and trajectories will still take off like $e^{\frac{3}{2}s}$.

To this end we estimate

$$\begin{aligned} |II| &\leq C_\alpha \int_{h \leq |\eta| \leq e^{ms}} \frac{e^{-s} + \varepsilon^{\frac{3}{4}\ell} \langle y \rangle^{-2/3}}{|y - \eta|^{1+2\alpha}} d\eta \\ &\leq 2C_\alpha (e^{-s} + \varepsilon^{\frac{3}{4}\ell} \langle h \rangle^{-2/3}) (e^{ms} - h) |y - e^{ms}|^{1+2\alpha}, \\ |III| &\leq 2C_\alpha \int_{e^{ms} \leq |\eta| < \infty} \frac{e^{-s}}{|y - \eta|^{1+2\alpha}} d\eta \\ &= \frac{2C_\alpha}{\alpha} e^{-s} |y - e^{ms}|^{-2\alpha} \end{aligned}$$

since $|y - e^{ms}| > 1$ our lower bound (3.23) applies (with $y_0 = \Phi(s_*) - e^{ms_*}$) and

$$|II^{y_0}(s)| \lesssim e^{ms} e^{-\frac{3}{2}(1+2\alpha)s}, \quad |III^{y_0}(s)| \lesssim e^{-(1+3\alpha)s}$$

for s sufficiently large. Collecting the estimates on I, II , and III proves the lemma. \square

We proceed in the same manner as we did when closing the bootstraps for \widetilde{W} on $h \leq |y| < \infty$ and $\partial_y \widetilde{W}$ on $h \leq |y| \leq e^{ms}$. Making the weighted change of variables $V := e^s \partial_y W$, a short computation using (C.1) shows that V satisfies the equation

$$(\partial_s + \beta_\tau \partial_y W) V + g_W \partial_y V = -\beta_\tau e^{3\alpha s} (-\Delta)^\alpha \partial_y W.$$

Closing the bootstrap (3.2c) now amounts to showing that $|V| < 2$ for all $e^{ms} \leq |y|$. We use (3.8) to bound the damping below

$$\mathcal{D}(y) = \beta_\tau \partial_y W \geq -\frac{3}{2} e^{-s}$$

In particular the damping is integrable and we obtain

$$\lambda_{\mathcal{D}} := e^{-\int_{s_*}^s \mathcal{D}^{y_0} ds'} \leq \varepsilon^{\frac{3}{2}}$$

Lemma 4.1 shows that the forcing is bounded and we applying our trajectory framework from Appendix A.1 to obtain

$$|V^{y_0}(s)| \lesssim \lambda_{\mathcal{D}} |V(y_0)| + \lambda_{\mathcal{D}} \int_{s_*}^s e^{(3\alpha-1)s'} ds'.$$

We once again have either $s_* = s_0$ or $s_* > s_0$. In the first case our initial data assumption (2.13c) gives

$$|V^{y_0}(s)| \lesssim \lambda_{\mathcal{D}} \left(e^{s_0} \varepsilon^2 2 + e^{(3\alpha-1)s_*} \right) \leq 1,$$

upon ε sufficiently small. In the second case our bootstrap on $h \leq |y| \leq e^{ms}$ (3.2b) implies that

$$|V^{y_0}(s)| \lesssim \lambda_{\mathcal{D}} \left(\langle e^{ms_*} \rangle^{-2/3} + \varepsilon^{\frac{3}{4}\ell} \langle e^{ms} \rangle^{-2/3} + e^{(3\alpha-1)s_*} \right) \leq 1,$$

since $e^{-ms_*} \leq e^{-ms_0} = \varepsilon^m$. The inequality holds upon choosing ε sufficiently small. This closes the bootstrap (3.2c).

5 Proof of Theorem

We are now ready to prove Theorem 2.1 and the Corollary 2.1.1. We prove uniform Hölder bounds on our solution up to the shock time in subsection 5.1, and asymptotic convergence to a stable Burgers profile Ψ_ν in subsection 5.2. We prove Theorem 2.1 in subsection 5.3 and we conclude with subsection 5.5 discussing why our method cannot be extended beyond the range $0 < \alpha < 1/3$.

5.1 Hölder Bounds

First we note that for all $0 \leq |y| \leq h < 1/2$, the bootstrap (3.2a) gives

$$|\widetilde{W}| \leq \left(\varepsilon^{\frac{8}{9}(1-3\alpha)} + \log M \varepsilon^{\frac{7}{9}(1-3\alpha)} \right) h^3 \int_0^{|y|} 1 dy = Ch^4 |y| \leq Ch^4 |y|^{1/3}.$$

Combining this estimate with the bootstrap (3.1b) we get

$$|\widetilde{W}| \leq \begin{cases} Ch^3 |y|^{1/3}, & 0 \leq |y| \leq h \\ \varepsilon^q \langle y \rangle^{1/3}, & h \leq |y| < \infty \end{cases} \leq |y|^{1/3}.$$

We then note that the bound (B.7) for Ψ implies that

$$|W| \leq |\widetilde{W}| + |\Psi| \leq 2|y|^{1/3}. \quad (5.1)$$

From the transformation (2.3) we obtain the following equality for the Hölder seminorms of u and W

$$\begin{aligned} [u]_{C^{1/3}} &= \sup_{\substack{x, z \in \mathbf{R} \\ x \neq z}} \frac{|u(x, t) - u(z, t)|}{|x - z|^{1/3}} = \sup_{\substack{x, z \in \mathbf{R} \\ x \neq z}} \frac{e^{-s/2} |W(xe^{3s/2}, s) - W(ze^{3s/2}, s)|}{(e^{-3s/2} |xe^{3s/2} - ze^{3s/2}|)^{1/3}} \\ &= \sup_{\substack{x', z' \in \mathbf{R} \\ x' \neq z'}} \frac{|W(x', s) - W(z', s)|}{|x' - z'|^{1/3}} = [W]_{C^{1/3}}. \end{aligned}$$

Furthermore, (5.1) implies that $[W]_{C^{1/3}} \leq 2$ uniformly in both x and s , and since $u \in L^\infty$ we have that

$$\|u\|_{C^{1/3}} = \|u\|_{L^\infty} + [u]_{C^{1/3}} \leq M + 2. \quad (5.2)$$

Therefore u is Hölder $1/3$ uniformly in x and t . We also point out that by a similar argument, any Hölder norm larger than $1/3$ cannot be uniformly controlled in time and blows up when the singularity forms. Indeed, letting $\beta > 1/3$, a simple calculation shows that

$$[u]_{C^\beta} = e^{\frac{8}{9}(3\beta-1)} [W]_{C^\beta}.$$

Hence u is not C^β at T_* for any $\beta > \frac{1}{3}$.

5.2 Asymptotic Convergence to Stationary Solution

The key observation underpinning our analysis is that whenever $(-\Delta)^\alpha W$ is bounded the self-similar equation (2.4) governing the evolution of W formally converges to the self-similar Burgers equation (B.1). This section justifies this limit rigorously.

We follow the proof outlined by Yang in [Yan20], which is based on the proof in [BSV19a].

5.2.1 Taylor Expansion

Set $\nu = \lim_{s \rightarrow \infty} \partial_y^3 W(0, s)$; this limit exists by the fundamental theorem of calculus and the estimate (4.6). By (3.5), we know that $5 \leq |\nu| \leq 7$. Since Ψ_ν satisfies the same constraints at the origin as Ψ , namely (2.6) (c.f. Appendix B), we automatically obtain $W(0, s) = \Psi_\nu(0)$, $\partial_y W(0, s) = \Psi'_\nu(0)$, and $\partial_y^2 W(0, s) = \Psi''_\nu(0)$ for all $s \geq s_0$.

The difference $\widetilde{W}_\nu := W - \Psi_\nu$ (Ψ_ν defined in (B.6)) satisfies the evolution equation

$$\left(\partial_s - \frac{1}{2} + \Psi'_\nu \right) \widetilde{W}_\nu + \left(\frac{3}{2} y + W \right) \partial_y \widetilde{W}_\nu = -\beta_\tau F_{\widetilde{W}_\nu}, \quad (5.3)$$

where the forcing $F_{\widetilde{W}_\nu}$ is given below by (5.11). We will now show that $W(y, s) \rightarrow \Psi_\nu(y)$ for all $y \in \mathbf{R}$ as $s \rightarrow \infty$, that is

$$\limsup_{s \rightarrow \infty} |\widetilde{W}_\nu(y, s)| = 0. \quad (5.4)$$

The convergence (5.4) is trivial when $y_0 = 0$ since $\Psi_\nu(0) = \Psi(0)$ and $W(0, s) = 0$ by our constraint (2.6).

We consider the Taylor expansion of \widetilde{W}_ν , observing that this Taylor expansion vanishes to third order as a consequence of our constraints (2.6):

$$\begin{aligned} \widetilde{W}_\nu(y, s) &= \widetilde{W}_\nu(0, s) + y \partial_y \widetilde{W}_\nu(0, s) + \frac{y^2}{2} \partial_y^2 \widetilde{W}_\nu(0, s) + \frac{y^3}{6} \partial_y^3 \widetilde{W}_\nu(0, s) + \frac{y^4}{24} \partial_y^4 \widetilde{W}_\nu(\eta, s) \\ &= \frac{y^3}{6} \partial_y^3 \widetilde{W}_\nu(0, s) + \frac{y^4}{24} \partial_y^4 \widetilde{W}_\nu(\eta, s), \quad 0 < |\eta| < \infty. \end{aligned}$$

Differentiating (B.6) four times, applying (3.18b), and taking ε small gives the following estimate

$$\begin{aligned} \|\partial_y^4 \widetilde{W}_\nu\|_{L_{y,s}^\infty} &\leq \|\Psi_\nu^{(4)}\|_{L^\infty} + \|\partial_y^4 W\|_{L_{y,s}^\infty} \\ &\leq 30 \left(\frac{7}{6} \right)^{\frac{3}{2}} + M^{\frac{6}{7}} \leq 2M^{\frac{6}{7}}. \end{aligned}$$

We take the absolute value of the Taylor expansion and apply (4.6) to find that

$$|\widetilde{W}_\nu(y, s)| \leq \frac{1}{6} |y|^3 e^{-\frac{1}{4}(3\alpha-1)s} + \frac{M^{\frac{1}{2}}}{12} |y|^4 \quad (5.5)$$

holds for all $y \in \mathbf{R}$.

Now fix any $y_0 \in \mathbf{R}$, with $|y_0| > 0$, and choose $0 < \lambda < 1$ such that $\lambda \leq |y_0| \leq \lambda^{-1/6}$. From the Taylor expansion (5.5), we may choose

$$s_* = \max \left\{ \log \left(\left(\frac{6\delta}{|\lambda|^3} \right)^{-\frac{4}{3\alpha-1}} \right), s_0 \right\}.$$

If we then choose some δ such that $\lambda^4 > \delta > 0$, then our Taylor expansion (5.5) collapses to

$$|\widetilde{W}_\nu(y, s)| \leq \delta + \frac{M^{\frac{1}{2}}}{12} |y|^4. \quad (5.6)$$

This choice for δ will become clear below.

5.2.2 Lagrangian Trajectories

Consider the Lagrangian flow associated with (5.3), given by

$$\frac{d}{ds}\Phi^{y_0} = \frac{3}{2}\Phi^{y_0} + W^{y_0}(s). \quad (5.7)$$

From the Taylor expansion of W about $y = 0$, the mean value theorem together with the uniform L^∞ bound (3.7) shows

$$|W(y, s)| \leq |y| \text{ for all } y \in \mathbf{R}.$$

Therefore $\frac{d}{ds}|\Phi^{y_0}|^2 \geq |\Phi^{y_0}|^2$, which upon integration yields the lower bound

$$|\Phi^{y_0}(s)| \geq |y_0|e^{\frac{1}{2}(s-s_0)}.$$

The same upper bound as before, (3.21), still applies to the current trajectories. Thus we have established that for all $0 < y_0 < \infty$

$$|y_0|e^{\frac{1}{2}(s-s_0)} \leq \Phi^{y_0}(s) \leq \left(|y_0| + \frac{3}{2}C\varepsilon^{-1/2}\right)e^{\frac{3}{2}(s-s_0)}. \quad (5.8)$$

Note that our new bounds are independent of any fixed parameter h , and indeed hold for any $y_0 > 0$. This is in contrast to our previous estimates, where some trajectories near the origin will not take off at all!

5.2.3 Forcing Terms

We prove two technical lemmas to deal with the fractional Laplacian in the range $1/4 \leq \alpha < 1/3$.

Lemma 5.1. *For $\alpha > 1/6$, we have*

$$|((-\Delta)^\alpha W)^0| \lesssim \frac{1}{h^{2\alpha}}. \quad (5.9)$$

Proof. Let χ be a smooth cutoff function such that $\chi \equiv 1$ on $0 < |y| < h/2$, $\chi \equiv 0$ on $h < |y|$, and which smoothly interpolates between 1 and 0 on $h/2 < |y| < h$ with $|\chi'| \leq 4/h$. We decompose

$$((-\Delta)^\alpha W)^0 = ((-\Delta)^\alpha(\chi W))^0 + ((-\Delta)^\alpha((1-\chi)W))^0$$

To estimate the first term, we use our interpolation (A.3) on $(-\Delta)^\alpha$, the bootstraps (3.1a) and (3.2a), and $|\chi'| \leq 4/h$ to get

$$\|(-\Delta)^\alpha(\chi W)^0\|_{L^\infty} \lesssim \|\chi W^0\|_{L^\infty}^{1-2\alpha} \|(\chi W)'\|_{L^\infty}^{2\alpha} \lesssim \frac{1}{h^{2\alpha}}.$$

To bound the second term, we use (1.2) and the fact that $2/3 + 2\alpha > 1$ to obtain

$$|(-\Delta)^\alpha[(1-\chi)W]^0| \leq 2C_\alpha \int_{h/2}^\infty (1-\chi(\eta)) \frac{|W(\eta)|}{\eta^{1+2\alpha}} d\eta \lesssim \int_{h/2}^\infty \eta^{-\frac{2}{3}-2\alpha} d\eta \lesssim h^{\frac{1}{3}-2\alpha}.$$

Since $h < 1$ the result follows. \square

Lemma 5.2. *For $\alpha \in (0, 1/3)$, we have*

$$\|(-\Delta)^\alpha W\|_{L^\infty} \lesssim \begin{cases} 1 + Me^{(\frac{1}{2}-3\alpha)s} & \alpha \neq 1/6 \\ M + s & \alpha = 1/6 \end{cases} \quad (5.10)$$

Proof. We decompose the domain of the singular integral representation (1.2) of $(-\Delta)^\alpha$ into three regions and estimate

$$|(-\Delta)^\alpha W(y)| \lesssim \int_{\mathbb{R}} \frac{|W(y) - W(\eta)|}{|y - \eta|^{1+2\alpha}} d\eta \lesssim \left(\int_{0 < |\eta - y| < 1} + \int_{1 < |\eta - y| < e^{3s/2}} + \int_{e^{3s/2} < |\eta - y|} \right) \frac{|W(y) - W(\eta)|}{|y - \eta|^{1+2\alpha}} d\eta.$$

Around y , the mean value theorem and (3.7) give $|W(y) - W(\eta)| \leq |y - \eta|$. Therefore

$$\int_{0 < |\eta - y| < 1} \frac{|W(y) - W(\eta)|}{|y - \eta|^{1+2\alpha}} d\eta \lesssim \int_{0 < |\eta - y| < 1} \frac{1}{|y - \eta|^{2\alpha}} d\eta \lesssim 1.$$

In the second region we use the *a posteriori* Hölder $C^{1/3}$ seminorm (5.1) to obtain

$$\int_{1 < |\eta-y| < e^{3s/2}} \frac{|W(y) - W(\eta)|}{|y - \eta|^{1+2\alpha}} d\eta \lesssim \int_{1 < |\eta-y| < e^{3s/2}} \frac{|y - \eta|^{1/3}}{|y - \eta|^{1+2\alpha}} d\eta \lesssim \begin{cases} 1 + e^{(\frac{1}{2}-3\alpha)s} & \alpha \neq 1/6 \\ 1 + s & \alpha = 1/6. \end{cases}$$

Finally, we use (3.10) and (3.9) to estimate the third integral

$$\int_{e^{3s/2} < |\eta-y|} \frac{|W(y) - W(\eta)|}{|y - \eta|^{1+2\alpha}} d\eta \lesssim M e^{s/2} \int_{e^{3s/2} < |\eta-y|} \frac{1}{|y - \eta|^{1+2\alpha}} d\eta \lesssim M e^{(\frac{1}{2}-3\alpha)s}.$$

The three estimates above prove the lemma. \square

We proceed to bound the forcing term in (5.3), which is given by

$$F_{\widetilde{W}_\nu} = \underbrace{e^{-s/2} \dot{\kappa}}_I + \underbrace{e^{(3\alpha-1)s} (-\Delta)^\alpha W}_{II} + \underbrace{\partial_y W e^{s/2} (\kappa - \dot{\xi})}_{III} + \underbrace{\dot{\tau} W \partial_y W}_{IV} \quad (5.11)$$

In what should now feel like a *danse familière*, we go term by term bounding the forcing in L^∞ .

When $\alpha < 1/4$, we use the identity (2.7b), the interpolation estimate (A.3), and the bootstrap (3.5) to obtain

$$\begin{aligned} |I| &\leq e^{(3\alpha-1)s} \left(\frac{|(-\Delta)^\alpha \partial_y^2 W^0(s)|}{|\partial_y^3 W^0(s)|} + |(-\Delta)^\alpha W^0(s)| \right) \\ &\lesssim M^{1/6+3\alpha} e^{(2\alpha-\frac{1}{2})s} \\ &\leq e^{(\alpha-\frac{1}{4})s} \end{aligned}$$

On the other hand, when $\alpha \geq 1/4$, we use Lemma 5.1 to handle the second term, and obtain

$$|I| \lesssim \frac{M^{1/6+3\alpha}}{h^{2\alpha}} e^{(3\alpha-1)s} \leq e^{\frac{1}{2}(3\alpha-1)s}.$$

Lemma 5.2 means that the second term is bounded by

$$|II| \lesssim e^{(3\alpha-1)s} \left(1 + M e^{(1/2-3\alpha)s} \right) \leq e^{-\frac{1}{4}s},$$

upon taking ε small.

The identity (2.7b), the interpolation estimate (A.3), the bounds (3.18c)-(3.17), the bootstrap (3.2), and the lower bound (5.8) on the trajectories gives us

$$|III| \lesssim (1 + \varepsilon^{\frac{3}{4}\ell}) \langle y \rangle^{-2/3} e^{(3\alpha-1)s} M^{\frac{2}{7}(1-\alpha)} \lesssim (1 + \varepsilon^{\frac{3}{4}\ell}) M^{\frac{2}{7}(1-\alpha)} e^{(3\alpha-\frac{4}{3})s}.$$

For the last term, the bootstraps (3.1)-(3.2), the lower bound (5.8) on our trajectories, (3.6b), and taking ε small gives us the bound

$$|IV| \leq (1 + \varepsilon^{\frac{1}{2}q})(1 + \varepsilon^{\frac{3}{4}\ell}) \langle y \rangle^{-\frac{1}{3}} \leq 2e^{-\frac{1}{6}s}.$$

To summarize, we have shown that

$$|F_{\widetilde{W}_\nu}^\lambda(s)| \lesssim e^{-ps}, \quad p = p(\alpha) > 0. \quad (5.12)$$

5.2.4 Putting it all Together

We set $G(y, s) = e^{-\frac{3}{2}(s-s_*)} \widetilde{W}_\nu(y, s)$, then compose with trajectories (5.7) and compute

$$\begin{aligned} \frac{d}{ds} G^\lambda &= -\frac{3}{2} G^\lambda + e^{-\frac{3}{2}(s-s_*)} \frac{d}{ds} \widetilde{W}_\nu^\lambda \\ &= -\frac{3}{2} G^\lambda + e^{-\frac{3}{2}(s-s_*)} \left[\left(\frac{1}{2} - (\Psi'_\nu)^\lambda \right) \widetilde{W}_\nu^\lambda - \beta_\tau F_{\widetilde{W}_\nu}^\lambda \right] \\ &= \left(-1 - (\Psi'_\nu)^\lambda \right) G^\lambda - \beta_\tau e^{-\frac{3}{2}(s-s_*)} F_{\widetilde{W}_\nu}^\lambda. \end{aligned}$$

Note that the damping $1 + (\Psi'_\nu)^\lambda \geq 0$ for all $s \geq s_*$, since $\|\Psi'_\nu\|_{L^\infty} = 1$ by (B.6). Thus we can apply Grönwall's inequality, our forcing estimate (5.12), and our Taylor expansion (5.6), and take s_* sufficiently large to obtain

$$\begin{aligned}
 |G^\lambda| &\leq |G(\lambda, s_*)| + \beta_\tau \int_{s_*}^s e^{-\frac{3}{2}(s'-s_*)} |F_{\widetilde{W}_\nu}^\lambda(s')| ds' \\
 &\lesssim |\widetilde{W}_\nu^\lambda(s_*)| + 2 \int_{s_*}^s e^{-\frac{3}{2}(s'-s_*)} e^{-ps'} ds' \\
 &\lesssim |\widetilde{W}_\nu^\lambda(s_*)| + e^{-ps_*} \\
 &\leq \delta + \frac{M^{\frac{1}{2}}}{12} |\lambda|^4 + \delta \\
 &\lesssim M^{\frac{1}{2}} |\lambda|^4.
 \end{aligned}$$

For all times $s_* \leq s \leq s_* + \frac{7}{3} \log |\lambda|^{-1}$, by our definition of G , we have that

$$\begin{aligned}
 |\widetilde{W}_\nu^{y_0}| &\lesssim M^{\frac{1}{2}} |\lambda|^4 e^{\frac{3}{2}(s-s_*)} \\
 &\lesssim M^{\frac{1}{2}} |\lambda|^{1/2}.
 \end{aligned}$$

For all y between λ and $\Phi(\lambda, s_* + \frac{7}{3} \log |\lambda|^{-1})$, there exists $s_* \leq s \leq s_* + \frac{7}{3} \log |\lambda|^{-1}$ such that $y = \Phi(\lambda, s)$. Therefore, for any pair (y, s) the previous estimate gives

$$|\widetilde{W}_\nu(y, s)| \lesssim M^{\frac{1}{2}} |\lambda|^{1/2}.$$

By composing with our trajectory lower bound this will cover at least all y such that

$$\lambda \leq |y| \leq \lambda e^{\frac{1}{2}(s-s_*)} = \lambda^{-1/6}.$$

Now, taking the limit that $s_* \rightarrow \infty$, for all $\lambda \leq |y| \leq \lambda^{-1/6}$ we have that

$$\limsup_{s \rightarrow \infty} |\widetilde{W}_\nu(y, s)| \lesssim M^{\frac{1}{2}} |\lambda|^{1/2}.$$

Finally, taking $\lambda \rightarrow 0$ proves that for all $y \neq 0$

$$\limsup_{s \rightarrow \infty} |\widetilde{W}_\nu(y, s)| = 0.$$

This completes the proof of (5.4).

5.3 Proof of Theorem 2.1

We are finally equipped to prove our main theorem, Theorem 2.1.

- (i) **(Solution is smooth before T_*).** Recall the uniform L^2 bound for $\partial_y W$ proven in Lemma 3.1 and the uniform L^2 bound of $\partial_y^6 W$ proven in (3.16); interpolation via Gagliardo-Nirenberg (Lemma A.2) bounds the intermediary L^2 norms of W uniformly in self-similar time.

From the transformation (2.3) and by differentiating through the L^2 norm we obtain the following family of identities

$$\|\partial_x^n u\|_{L^2} = e^{(-\frac{5}{4} + \frac{3}{2}n)s} \|\partial_y^n W\|_{L^2}$$

relating the L^2 norms in physical space to the L^2 norms in the self-similar space. Note that Burgers equation satisfies L^2 conservation and the fractional term may be dropped from estimates, ensuring that $\|u\|_{L^2}$ is uniformly bounded. These identities along with the uniform boundedness of the L^2 norms for $\partial_y W$ through $\partial_y^6 W$ shows that the H^6 norm of u remains finite for all times prior to T_* .

From the above considerations and the local-in-time existence proven in [KNS08, ADV07] we deduce that $u \in C([-\varepsilon, \bar{T}]; H^6(\mathbf{R}))$ for any $\bar{T} < T_*$.

(ii) **(Blowup location is unique)**. Fix $x^b \neq x_*$. Differentiating the transformation (2.3) gives the identity

$$\partial_x u(x^b, t) = e^s \partial_y W \left((x^b - \xi(t)) e^{\frac{3}{2}s}, s \right).$$

Because $x^b \neq x_*$ and $\xi(t) \rightarrow x_*$ as $t \rightarrow T_*$, there exists a time t^b such that for all $t^b < t \leq T_*$ we can choose $1 > \lambda > 0$ such that

$$|x^b - \xi(t)| > \lambda > 0.$$

Therefore

$$|x^b - \xi(t)| e^{\frac{3}{2}s} > \lambda e^{\frac{3}{2}s},$$

and for all

$$s > s_\lambda := \max \left\{ \frac{3}{2} \left(\frac{1}{3/2 - m} \right) \log \lambda^{-1}, -\log(T_* - t^b) \right\},$$

we satisfy the inequality

$$|x^b - \xi(t)| e^{\frac{3}{2}s} > e^{ms}.$$

We emphasize that the lower bound s_λ on times for which the above inequality holds grows arbitrarily large as $\lambda \rightarrow 0^+$.

We can apply our bootstrap (3.2c) to find that for all s in this range

$$|\partial_y W \left((x^b - \xi(t)) e^{\frac{3}{2}s}, s \right)| \lesssim e^{-s},$$

and hence

$$\limsup_{t \rightarrow T_*} |\partial_x u(x^b, t)| \lesssim 1.$$

Thus the blowup location is unique.

(iii) **(Blowup time and location)**. From our definition of τ , the blowup time T_* is the unique fixed point $\tau(T_*) = T_*$, and in light of (2.1) this is equivalent to

$$\int_{-\varepsilon}^{T_*} (1 - \dot{\tau}(t)) dt = \varepsilon.$$

Applying the bootstrap closure (4.1) we find that

$$\varepsilon = \int_{-\varepsilon}^{T_*} (1 - \dot{\tau}(t)) dt \geq \int_{-\varepsilon}^{T_*} 1 - \frac{3}{4} \varepsilon^{\frac{8}{9}(1-3\alpha)} dt,$$

from which it follows that $|T_*| \leq \frac{3}{4} \varepsilon^{\frac{8}{9}(1-3\alpha)}$ upon choosing ε sufficiently small.

Similarly, our definition of ξ together with (2.1) gives the condition

$$\int_{-\varepsilon}^{T_*} \dot{\xi}(t) dt = x_*.$$

Applying the bootstrap closure (4.2) gives us

$$x_* = \int_{-\varepsilon}^{T_*} \dot{\xi}(t) dt \leq 2M(T_* + \varepsilon) \leq 3M\varepsilon.$$

This proves the claimed bounds on T_* and y_* .

(iv) **(Precise control of $\partial_x u$ at $t = T_*$)**. u develops a shock (gradient blowup) at $t = T_*$. Recall the identity (2.2) which, together with differentiating the transformation (2.3), gives

$$\partial_x u(\xi(x), t) = \frac{1}{\tau(t) - t} \partial_y W(0, s) \geq -\frac{1}{\tau(t) - t}.$$

Next note that for all $-\varepsilon \leq t < T_*$ we have that

$$\frac{1}{2} \leq \frac{\tau(t) - t}{T_* - t} \leq 1.$$

The upper bound is obvious since $\tau(t)$ is monotone increasing and T_* is the unique fixed point of τ . The lower bound is a consequence of the fact that

$$\frac{T_*}{2} \leq \tau(t) - \frac{1}{2}t.$$

This follows from the fact that $\tau(t) - t/2$ is monotone decreasing and at time $t = T_*$ the r.h.s. is $T_*/2$. We conclude that

$$-\frac{1}{\tau(t) - t} \leq \partial_x u(\xi(x), t) = -\|\partial_x u(\cdot, t)\|_{L^\infty} \leq -\frac{1}{2} \frac{1}{\tau(t) - t}.$$

Thus we have proven the desired behavior of the gradient at the singularity.

(v) ($W \rightarrow \Psi$ asymptotically in self-similar space). This was proved above in Section 5.2.

(vi) (**Shock is Hölder** $^{1/3}$). This is a consequence of the uniformity of the Hölder bound which was shown above in Section 5.1. Take the limit as $s \rightarrow \infty$ to find that $u(y, T_*) \in C^{1/3}$.

5.4 Proof of Corollary 2.1.1

We begin by noting that κ_0 and ε can be taken in an open neighborhood since all of our previous arguments require only that ε is sufficiently small. From (2.10), this implies that $u_0(0)$ and $\partial_x u_0(0)$ can be taken in an open set.

Interpolating between (2.21b) and (2.21c) yields $\|\partial_x^4 u_0\|_{L^\infty} = \mathcal{O}(\varepsilon^{-11/2})$. Using this in a Taylor expansion for $\partial_x^2 u_0$ around the origin, we find that for all $0 < |\bar{x}| < |x|$

$$\begin{aligned} \partial_x^2 u_0(x) &= \partial_x^2 u_0(0) + \partial_x^3 u_0(0)x + \frac{\partial_x^4 u_0(\bar{x})}{2} \bar{x}^2 \\ &= \partial_x^2 u_0(0) + 6\varepsilon^{-4}x + x(\partial_x^3 u_0(0) - 6\varepsilon^{-4}) + \mathcal{O}(\varepsilon^{-11/2})\bar{x}^2. \end{aligned}$$

We will show that we can relax $\partial_x^2 u_0(0) = 0$ and (2.17) by assuming instead that $|\partial_x^2 u_0(0)|$ is sufficiently small, and considering a small interval around 0 for x on which $|\partial_x^3 u_0(x) - 6\varepsilon^{-4}| < 7\varepsilon^{-4/\beta(5+12\alpha)}$. We can take $x(\partial_x^3 u_0(0) - 6\varepsilon^{-4})$ small relative to $6\varepsilon^{-4}x$, and similarly for $\mathcal{O}(\varepsilon^{-11/2})\bar{x}^2$. Since $6\varepsilon^{-4}x$ dominates, by intermediate value theorem, there exists x^* sufficiently small so $\partial_x^2 u_0(x^*) = 0$. We can now simply change coordinates $x \mapsto x + x^*$.

The inequalities in (2.18)-(2.21c) can be replaced by strict inequalities by introducing a pre-factor slightly greater than 1. A sufficiently small H^6 perturbation preserves all these inequalities by slightly enlarging the pre-factor. In particular, by Sobolev embedding, a small $W^{5,\infty}$ perturbation will satisfy these inequalities.

5.5 Stable Modulation Past $\alpha = 1/3$

Since the stable Burgers profile satisfies $-1 < \Psi'(x) < 0$ for $x \neq 0$ with $\Psi'(0) = -1$, it follows from (1.2) that $(-\Delta)^\alpha[\Psi'](0) < 0$. If we consider W to be a perturbation of Ψ , the modulation constraint (2.7a) for τ amounts to

$$\dot{\tau} \approx e^{(3\alpha-1)s} |(-\Delta)^\alpha[\Psi'](0)|. \quad (5.13)$$

When $\alpha \geq 1/3$, the right-hand side is positive and bounded away from zero for all s , so the modulation parameter τ diverges as $s \rightarrow \infty$.

Oh and Pasqualotto [OP21] have recently proven that the solutions to fractal Burgers equation (1.1) in the range $1/3 \leq \alpha < 1/2$ with specifically chosen initial data converge in an asymptotically self-similar manner to *unstable* profiles. However these solutions are not stable under *generic* perturbations.

The blowup criterion supplied by Dong, Du, and Li [DDL09] is continuous which means that there is stable singularity formation in the range $1/3 \leq \alpha < 1/2$. Our numerics suggest that this stable blowup profile is not the stable Burgers profile.

A Toolbox

A.1 A Framework for Weighted Transport Estimates

We present a framework for doing weighted transport estimates which was originally introduced in [BSV19a]. See also Section 6.5 of [Yan20]. We sketch the main ideas here. Our goal is to bound the L^∞ evolution of a transport-type equation by using information over some spatial region.

Consider the forced-damped transport equation

$$\partial_s \mathcal{U} + \mathcal{D}\mathcal{U} + g_W \partial_y \mathcal{U} = \mathcal{F}.$$

We denote the damping \mathcal{D} and the forcing \mathcal{F} . The advection velocity g_W is defined in (2.5).

We consider the weighted quantity $V := \langle x \rangle^p \mathcal{U}$, where p is some power. It is straightforward to compute the evolution of V

$$\partial_s V + (\mathcal{D} - 2g_W p x \langle x \rangle^{-1} g_W) V + g_W V = \langle x \rangle^p \mathcal{F}. \quad (\text{A.1})$$

Consider the Lagrangian trajectory Φ^{x_0} obtained above by solving (3.20), and set $V^{x_0}(s) := V \circ \Phi^{x_0}(s)$. Composing the evolution (A.1) with $\Phi^{x_0}(s)$ gives

$$\frac{d}{ds} V^{x_0} + (\mathcal{D} \circ \Phi^{x_0} - 2g_W p \Phi^{x_0} \langle \Phi^{x_0} \rangle^{-1} g_W) V^{x_0} + g_W V^{x_0} = \langle \Phi^{x_0} \rangle^p \mathcal{F} \circ \Phi^{x_0}.$$

This equation can be solved via separation of variables to obtain

$$V^{x_0}(s) = V(x_0) e^{-\int_{s_*}^s \mathcal{D} \circ \Phi^{x_0}(s') ds'} + \int_{s_*}^s \mathcal{F} \circ \Phi^{x_0}(s') e^{-\int_{s_*}^{s'} \mathcal{D} \circ \Phi^{x_0}(s'') ds''} ds', \quad (\text{A.2})$$

where s_* is the first time that the trajectory enters the region in question.

A.2 Lemmas

We make frequent use of the following lemma to bound the fractional Laplacian in terms of our L^∞ bootstraps.

Lemma A.1. (*L^∞ Interpolation of the Fractional Laplacian.*) *Let $u \in W^{1,\infty}(\mathbf{R})$ and $\alpha \in (0, 1/2)$, then $(-\Delta)^\alpha u \in L^\infty$ and satisfies the following estimate*

$$\|(-\Delta)^\alpha u\|_{L^\infty} \lesssim \|u\|_{L^\infty}^{1-2\alpha} \|u'\|_{L^\infty}^{2\alpha}. \quad (\text{A.3})$$

Proof. Let $0 < h$ to be chosen later and write the fractional Laplacian in its singular integral form (1.2)

$$\begin{aligned} \frac{1}{C_\alpha} (-\Delta)^\alpha u &= \int_{\mathbf{R} \setminus \mathbf{B}_h(x)} \frac{u(x) - u(y)}{|x - y|^{2\alpha+1}} dy + \int_{\mathbf{B}_h(x)} \frac{u(x) - u(y)}{|x - y|^{2\alpha+1}} dy \\ &=: I_1 + I_2. \end{aligned}$$

For the first integral we estimate

$$\begin{aligned} |I_1| &\leq \int_{\mathbf{R} \setminus \mathbf{B}_h(x)} \frac{|u(x) - u(y)|}{|x - y|^{1+2\alpha}} dy \leq 4\|u\|_{L^\infty} \int_h^\infty \frac{1}{y^{1+2\alpha}} dy \\ &\leq \frac{1}{\alpha h^{2\alpha}} \|u\|_{L^\infty}. \end{aligned}$$

For the second integral, the mean value theorem gives

$$u'(\eta, t) = \frac{u(x, t) - u(y, t)}{x - y}, \quad |\eta| \leq |x - y|.$$

This implies that

$$\begin{aligned} |I_2| &\leq \int_{\mathbf{B}_h(x)} \frac{|u(x) - u(y)|}{|x - y|^{1+2\alpha}} dy = \int_{\mathbf{B}_h(x)} \frac{|u'(\eta)|}{|x - y|^{2\alpha}} dy \\ &\leq 2\|u'\|_{L^\infty} \int_0^h \frac{1}{y^{2\alpha}} dy \\ &= \frac{1}{1 - 2\alpha} h^{1-2\alpha} \|u'\|_{L^\infty}. \end{aligned}$$

To summarize, we now have

$$\frac{1}{C_\alpha}(-\Delta)^\alpha u \leq \frac{1}{\alpha h^{2\alpha}} \|u\|_{L^\infty} + \frac{1}{1-2\alpha} h^{1-2\alpha} \|u'\|_{L^\infty}.$$

Letting $A = \|u\|_{L^\infty}$, $B = \|u'\|_{L^\infty}$, and $h = A^\beta B^\gamma$, we see that the only possible scaling is $\beta = 1, \gamma = -1$, i.e. we choose $h = \|u\|_{L^\infty} / \|u'\|_{L^\infty}$ and obtain

$$\frac{1}{C_\alpha}(-\Delta)^\alpha u \leq \left(\frac{1}{\alpha} + \frac{1}{1-2\alpha} \right) \|u\|_{L^\infty}^{1-2\alpha} \|u'\|_{L^\infty}^{2\alpha}.$$

□

We also use the Gagliardo-Nirenberg Interpolation Theorem for \mathbf{R} .

Lemma A.2. (Gagliardo-Nirenberg Interpolation). Fix $1 \leq q, r \leq \infty$, $j, m \in \mathbf{N}$ with $j/m \leq \theta \leq 1$. If we have the following relationship

$$\frac{1}{p} = j + \theta \left(\frac{1}{r} - m \right) + \frac{1-\theta}{q},$$

then

$$\|\partial_x^j u\|_{L^p} \lesssim \|\partial_x^m u\|_{L^r}^\theta \|u\|_{L^q}^{1-\theta}.$$

The implicit constant depends only on j, m, r, p, q , and θ (i.e. is independent of u).

B Derivation and Properties of the Self-Similar Burgers Profile Family

Consider the similarity equation for the Burgers equation [EF15]

$$-\frac{1}{2}\Psi + \left(\frac{3}{2}y + \Psi \right) \Psi' = 0. \quad (\text{B.1})$$

This is a homogeneous ODE in Ψ . It is elementary to obtain the following implicit family of solutions

$$x = -\Psi_\nu - \frac{\nu}{6} \Psi_\nu^3, \quad (\text{B.2})$$

where $\nu/6$ is the constant of integration. We denote by Ψ_ν the solution of (B.1) corresponding to a particular choice of ν .

Taking successive (implicit) derivatives we easily obtain the following properties of Ψ_ν :

$$\Psi_\nu(0) = 0, \quad \Psi'_\nu(0) = -1, \quad \Psi''_\nu(0) = 0, \quad \Psi'''_\nu(0) = \nu, \quad \Psi_\nu^{(2k)}(0) = 0, \quad k \in \mathbf{N}. \quad (\text{B.3})$$

This shows that the solution family (B.2) is parameterized by it's third derivative at the origin.

We take by convention $\Psi := \Psi_6$, which has the explicit solution

$$\Psi(x) = \left(-\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4} \right)^{1/2} \right)^{1/3} - \left(\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4} \right)^{1/2} \right)^{1/3}. \quad (\text{B.4})$$

We use this particular profile throughout our analysis. Furthermore we can Taylor expand the derivative Ψ' around $x = 0$ to obtain

$$\Psi'(x) = -1 + 3x^2 - 15x^4 + \mathcal{O}(x^6). \quad (\text{B.5})$$

In fact, once we have obtained the solution for Ψ , the solutions for all of the profiles are obtained through the formula

$$\Psi_\nu(x) = \left(\frac{\nu}{6} \right)^{-1/2} \Psi \left(\left(\frac{\nu}{6} \right)^{1/2} x \right). \quad (\text{B.6})$$

This can be verified by noting that if Ψ solves (B.2) for $\nu = 1$, then Ψ_ν solves the same equation with, excusing the abuse of notation, $\nu = \nu$.

We note that the explicit form of *any* of the solutions in the family (B.2) can be obtained quite easily using Mathematica or some other solver (or by hand if the reader is a spiritual medium for Gerolamio Cardano), since the explicit solution (B.4) is nothing more than the solution of the cubic $x = -y - \nu y^3$.

Once an explicit solution is obtained, the reader can easily verify the following inequalities which (we suggest using Mathematica)

$$|\Psi(x)| \leq |x|^{1/3}, \quad (\text{B.7})$$

along with the Japanese bracket estimates

$$|\partial_x^i \Psi(x)| \lesssim \langle x \rangle^{1/3-i}, \quad (\text{B.8})$$

for $i = 1, \dots, 5$ and $x > 1$. In the case of $i = 1, 2$ the constant can be taken to be 1. The constant for $i = 5$, for example, can be taken as 360.

The fractional Laplacian applied to the stable profile $\Psi^{(n)}$ behaves exactly as one would expect, namely we have the following pointwise estimate:

Lemma B.1. *The bounds (B.8) imply*

$$|(-\Delta)^\alpha \Psi'(x)| \lesssim \langle x \rangle^{-2/3-2\alpha}. \quad (\text{B.9})$$

Proof. Without loss of generality, suppose $x > 0$. We decompose the integral form (1.2) of $(-\Delta)^\alpha$ by

$$(-\Delta)^\alpha \Psi'(x) = C_\alpha \left(\int_{-\infty}^{-x} + \int_{-x}^{x/2} + \int_{x/2}^x + \int_x^{2x} + \int_{2x}^{\infty} \right) \frac{\Psi'(x) - \Psi'(y)}{|x-y|^{1+2\alpha}} dy.$$

The first integral can be bounded as

$$\int_{-\infty}^{-x} \frac{|\Psi'(x) - \Psi'(y)|}{|x-y|^{1+2\alpha}} dy \lesssim \langle x \rangle^{-2/3} \int_{-\infty}^{-x} \frac{1}{|x-y|^{1+2\alpha}} dy \lesssim \langle x \rangle^{-2/3-2\alpha}.$$

The second integral can be bounded with

$$\int_{-x}^{x/2} \frac{|\Psi'(x) - \Psi'(y)|}{|x-y|^{1+2\alpha}} dy \lesssim \int_{-x}^{x/2} \frac{x^{-2/3} + |y|^{-2/3}}{x^{1+2\alpha}} dy \lesssim x^{-2/3-2\alpha} + \frac{1}{x^{1+2\alpha}} \int_{-x}^{x/2} y^{-2/3} dy \lesssim \langle x \rangle^{-2/3-2\alpha}.$$

To bound the third integral, we use the Holder seminorm bound

$$|\Psi'|_{C^{3\alpha}(x/2,x)} \leq \|\Psi'\|_{L^\infty(x/2,x)}^{1-3\alpha} \|\Psi''\|_{L^\infty(x/2,x)}^{3\alpha} \lesssim (\langle x \rangle^{-2/3})^{1-3\alpha} (\langle x \rangle^{-5/3})^{3\alpha} = \langle x \rangle^{-2/3-3\alpha}.$$

Using this seminorm bound, we obtain that the third integral can be estimated as

$$\lesssim |\Psi'|_{C^{3\alpha}(x/2,x)} \int_{x/2}^x \frac{1}{|x-y|^{1-\alpha}} dy \lesssim \langle x \rangle^{-2/3-3\alpha} x^\alpha \lesssim \langle x \rangle^{-2/3-2\alpha}.$$

The fourth integral is estimated similarly to the second integral, and the fifth integral is estimated similarly to the first. \square

C Evolution Equations For Derivatives and Differences

We record the equations which are used above in the course of our proof. The equations are written in the form $(\partial_s + D)f + g_W f_y = Ff$ where D is the damping and F the forcing.

We have the following differentiated forms of equation (2.4)

$$\left(\partial_s + 1 + \beta_\tau \partial_y W \right) \partial_y W + g_W \partial_y^2 W = -\beta_\tau e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y W \quad (\text{C.1})$$

$$\left(\partial_s + \frac{5}{2} + 3\beta_\tau \partial_y W \right) \partial_y^2 W + g_W \partial_y^3 W = -\beta_\tau e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^2 W \quad (\text{C.2})$$

$$\left(\partial_s + 4 + 4\beta_\tau \partial_y W \right) \partial_y^3 W + g_W \partial_y^4 W = -\beta_\tau \left[e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^3 W + 3(\partial_y^2 W)^2 \right] \quad (\text{C.3})$$

$$\left(\partial_s + \frac{11}{2} + 7\beta_\tau \partial_y W \right) \partial_y^4 W + g_W \partial_y^5 W = -\beta_\tau \left[e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^4 W + 35\partial_y^3 W \partial_y^2 W + 21\partial_y^2 W \partial_y^3 W \right] \quad (\text{C.4})$$

And the equations for the differences \widetilde{W} , $\partial_y \widetilde{W}$, and $\partial_y^4 \widetilde{W}$ derivative

$$\left(\partial_s - \frac{1}{2} + \beta_\tau \Psi\right) \widetilde{W} + g_W \partial_y \widetilde{W} = -\beta_\tau \left[e^{-\frac{s}{2}} \dot{\kappa} + e^{(3\alpha-1)s} (-\Delta)^\alpha W + \Psi'(\dot{\tau} \Psi + e^{\frac{s}{2}}(\kappa - \dot{\xi})) \right] \quad (\text{C.5})$$

$$\begin{aligned} & \left(\partial_s + 1 + \beta_\tau (\partial_y \widetilde{W} + 2\Psi')\right) \partial_y \widetilde{W} + g_W \partial_y^2 \widetilde{W} \\ & = -\beta_\tau \left[e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y W + \left(e^{s/2}(\kappa - \dot{\xi}) + \widetilde{W} + \dot{\tau} \Psi \right) \Psi'' + \dot{\tau} (\Psi')^2 \right] \end{aligned} \quad (\text{C.6})$$

$$\begin{aligned} \left(\partial_s + \frac{11}{2} + 5\beta_\tau \partial_y W\right) \partial_y^4 \widetilde{W} + g_W \partial_y^4 \widetilde{W} & = -\beta_\tau \left[e^{(3\alpha-1)s} (-\Delta)^\alpha \partial_y^4 W + \left(e^{s/2}(\kappa - \dot{\xi}) + \widetilde{W} + \dot{\tau} \Psi \right) \partial_y^5 \Psi \right. \\ & \quad + 4\partial_y \widetilde{W} \partial_y^4 \Psi + 8\partial_y^2 \widetilde{W} \partial_y^3 \Psi + 10\partial_y^3 \widetilde{W} \partial_y^2 \Psi \\ & \quad \left. + 11\partial_y^2 \widetilde{W} \partial_y^3 \widetilde{W} + 5\dot{\tau} \partial_y \Psi \partial_y^4 \Psi + 10\dot{\tau} \partial_y^2 \Psi \partial_y^3 \Psi \right] \end{aligned} \quad (\text{C.7})$$

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