Final Solutions:

1a) A inv $\iff$ $\det A \neq 0$

$$\det(A^k) = (\det A)^k \neq 0 \qquad \square$$

1b) A inv $\overset{(1b)}{\iff}$ A^k inv $\iff$ $\det(A^k) \neq 0$

$$1 = \det\left(A^k (A^k)^{-1}\right) = \det(A^k) \det(A^k)^{-1}$$

$$= (\det A)^k \det(A^k)^{-1}$$

$$\Rightarrow \quad \det(A^k)^{-1} = \frac{1}{(\det A)^k} \qquad \square$$

1c) See practice final $\square$

1d $\det(A) = \det(PJP^{-1})$

P is invertible

$$= \det(P)\det J \det(P^{-1})$$

$$= \frac{\det P}{\det P} \det J = \det J \qquad \square$$

2) Find $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

* You can also use diagonalization!

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad a + 2b = 1$$
$$c + 2d = 2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$-a + 2b = 2$$
$$-c + 2d = -4$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ -1 & 2 & 0 & 0 & 2 \\ 0 & 0 & -1 & 2 & -4 \end{array}\right)$$

$$R_4 = R_4 + R_2 \quad R_3 = R_3 + R_1 \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 4 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 & -2 \end{array}\right)$$

$$R_2 \leftrightarrow R_3 \quad \tfrac{1}{4}R_2 \quad \tfrac{1}{4}R_4 \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right)$$

$$R_1 = R_1 - 2R_2 \quad R_3 = R_3 - 2R_4 \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right)$$

So $A = \begin{pmatrix} -1/2 & 3/4 \\ 3 & -1/2 \end{pmatrix}$

$\blacksquare$

3a) We require

$$x^2 + y^2 = 0$$
$$z = 0$$
$$x + z = 0$$
$$y^2 + z^3 = 0$$

$z = 0 \Rightarrow$ $\left[\begin{array}{l} x = 0 \\ y^2 = 0 \end{array}\right.$ $\Rightarrow x^2 + y^2 = 0$

So $\ker f = \{(0,0,0)\}$ $\square$

3b) $f(x,y,z) = (\alpha, \beta, \gamma, \delta)$

$$= (x^2 + y^2,\ z,\ x + z,\ y^2 + z^3)$$

$\Rightarrow \beta = z, \quad \alpha = x^2 + y^2$

$$x = \sqrt{\alpha - y^2}$$

$\Rightarrow \gamma = \sqrt{\alpha - y^2} + \beta$

$\Rightarrow \delta = \alpha - x^2 + \beta^3$

$$x = \sqrt{\alpha - y^2} = \gamma - \beta$$

$$x^2 = \gamma^2 - 2\gamma\beta + \beta^2$$

so $$z = \alpha - \gamma^2 + 2\gamma\beta + \beta^2 + \beta^3$$

So

$$\operatorname{Rng} f = \{ (\alpha, \beta, \gamma, z) \in \mathbb{R}^4 \mid z - \alpha + \gamma^2 - 2\gamma\beta - \beta^2 - \beta^3 = 0 \}$$

$\square$

3c) f is not 1-1,

$$f(0,1,0,0) = (1,0,0,1)$$
$$f(0,-1,0,0) = (1,0,0,1)$$

not onto since $(-1,0,0,0)$ not in range. $\blacksquare$

4) We look at how

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \vdots & & \vdots & & \vdots \\ a_{51} & \cdots & & & a_{55} \end{pmatrix} \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}$$

behaves

$a_{11} = -1$, $a_{12} = 1$, ...

$$\Rightarrow$$

$$f = A = \begin{pmatrix} -1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

$\xrightarrow{RREF} \quad I$

so rank $A = 5$ & by rank-nullity,

$\dim(N(A)) = 0$. $\blacksquare$

5a) We look for functions such that

$$\frac{d}{dx} y = \lambda y \quad \text{which is solved by}$$

$$y = c e^{\lambda x}$$

So the eigenvalues are $\lambda \in \mathbb{R}$, since for any $\lambda \in \mathbb{R}$, $e^{\lambda x}$ is a solution, hence an eigenvector. $\square$

5b) We have only seen transforms on finite dimensional spaces, which will have finitely many eigenvalues.

We have here infinitely many eigenvalues where as normally we have seen finitely many. $\blacksquare$

6) Since $\mathbb{R}[a,b]$ a v.s., the subspace theorem says it is sufficient to prove closed under $+$ & $\times$:

(+) : f, g have zero mean

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$= 0 + 0 = 0$$

(x) : $$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx$$

$$= \alpha\, 0 = 0$$

$\blacksquare$

7a) No such f exists since the dimension of the domain is smaller than dim of range. $\square$

7b) No such f exists since the dimension of the domain is larger than dim of range. $\square$

7c) Not possible, $|S| < \dim M_{3\times 2}$ $\square$

7d) Not possible, $|S| > \dim M_{2\times 2}$ $\square$

7e) This can happen. Rank-Nullity says

$$\underbrace{\operatorname{rank} A}_{\text{at most } d} + \dim(N(A)) = m$$

, so if $m \geq 2d$ this is possible $\square$

7f) Not possible. To be similar to a diagonal matrix, must have

alg mult = geo mult.

$\blacksquare$

8a) Since $C^1(\mathbb{R})$ is a vector space, we must show the set of solutions is closed under $+$ and $\times$. Then the subspace Thm allows us to conclude set of sols is itself a v.s.

$\oplus$ Let u, w solve $\frac{d}{dx}u + u = 0$, $\frac{d}{dx}w + w = 0$

then $\frac{d}{dx}(u+w) + (u+w)$

$$= \frac{d}{dx}u + u + \frac{d}{dx}w + w = 0$$

$\otimes$ Let $\alpha \in \mathbb{R}$

$$\frac{d}{dx}(\alpha u) + (\alpha u)$$

$$= \alpha\left(\frac{d}{dx}u + u\right) = 0$$

So the set of sols is a v.s. $\square$

8b) Since e^{-x} is a solution, the space of solutions must be at least dimension 1, since $y \in \langle e^{-x} \rangle$ solves ODE.

8c) $\frac{d}{dx} y = -y$

$$\Rightarrow -\int \frac{1}{y} dx = x + c$$

$$\Rightarrow -\log|y| = x + c$$

$$\underline{y = c_0 e^{-x}}$$

All solutions look like this, i.e. a span

Since set of sols <u>is</u> $\langle e^{-x} \rangle$ and hence has dimension 1.

9a) Property 1:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad a = 1$$

property 2:

$$\underbrace{\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ c & cb \end{pmatrix}$$

$$\Rightarrow cb = d$$

property 3:

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ c & cb \end{pmatrix}\right]^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ c & cb \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}\right]^T = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \Rightarrow b = 0$$

property 4:

$$\left[\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]^T = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}\right]^T = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$$

$$\Rightarrow c = 0$$

So $\boxed{A = A^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ $\square$

9b) This property does not actually hold in general! $\square$

9c) $(A^+)^+ = A$

follows from properties 2 & 1, Let $A^+ = B$, then by 1

$$B(B^+)B = B$$

and by 2

$$A^+ A A^+ = BAB = B$$

$$\Rightarrow \quad A = B^+ = (A^+)^+$$

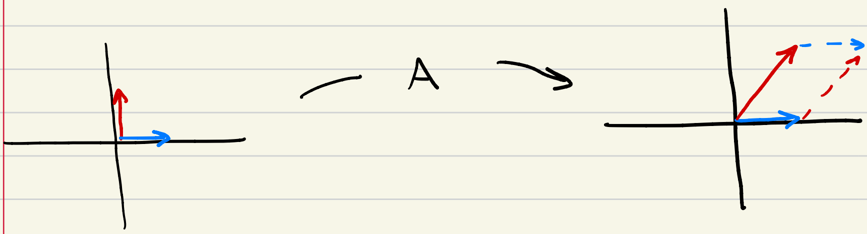
$\square$

9d) If A is invertible, all four properties are satisfied, since A^+ is unique, $A^+ = A^{-1}$.

$\blacksquare$

10a) We know $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

So



So this is a shear w/ a slight vertical stretch.

10b)

$$(A^2 - 2\mu A)w = -\mu^2 w$$

$$\Rightarrow (A^2 - 2\mu A + \mu^2 I)w = 0$$

$$\Rightarrow (A - \mu I)^2 w = 0$$

$$\det((A-\mu I)^2) = (\det(A-\mu I))^2 = 0$$

So $\det(A - \mu I) = 0$

$$\Rightarrow (1-\mu)(2-\mu) = 0 \Rightarrow \mu = 1, 2$$

$$(A-I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b = 0$$

choose $a = 1$, then

$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ solves for $\mu = 1$

(any multiple of w also works)

$$(A-2I)^2 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = b$$

$w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any multiple) for $\mu = 2$

$\square$

3c) These are the eigenvectors / eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ since they satisfy the same char polynomial, and the same eigenvector eqs. $\square$

3d) $A(A^2 + 3\mu^2 I)w - \mu(3A^2 + \mu^2 I)w$

$= (A^3 - 3\mu A^2 + 3\mu^2 A - \mu^3 I)w$

$= (A - \mu I)^3 w$

So true for $\mu = 1, 2$ and

$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (scalar mult)

by (a), (b) $\blacksquare$

(All eqs are the same)