22A WHW1 Solutions

1: (a)

$A \oplus B \neq B \oplus A$, consider

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then $A \oplus B = A$ but $B \oplus A = A^T$

and $A \neq A^T$ $\square$

(b)

$$A \oplus (B \oplus C) = A + (B + C^T)^T$$

$$= A + B^T + C$$

$$= (A \oplus B) + C$$

$$\neq (A \oplus B) \oplus C$$

if $C \neq C^T$. $\square$

(c)

Must find $-A \in M_{4\times 4}(\mathbb{R})$ so that

$$A \oplus (-A) = 0$$

$$A \oplus (-A) = A + (-A)^T$$

using $(A^T)^T = A$ and $(A+B)^T = A^T + B^T$

$$= (A^T + (-A))^T = 0^T = 0$$

so $\boxed{(-A) = -A^T}$ ■

2: (a)

$$(A+B)^T = (a_{ij} + b_{ij})_{m\times n}^T = (a_{ji} + b_{ji})_{m\times n}$$

$$= A^T + B^T$$ □

(b) No, if $m \neq n$, for any $A, B \in M_{m\times n}(\mathbb{R})$, $A \oplus B \notin M_{m\times n}(\mathbb{R})$, so not closed.

However, if $m = n$, then $M_{m\times n}(\mathbb{R})$ will be closed under $\oplus$, since $A \oplus B \in M_{m\times m}(\mathbb{R})$. ■

3: (a)

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\neq$

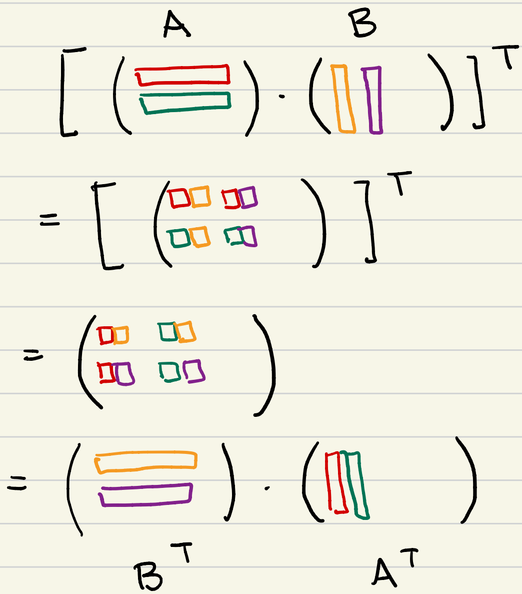
$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\square$

(b)

$$(AB)^T = B^T A^T$$

schematically:



Formally:

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{i,j}$$

$$\Rightarrow (AB)^T = \left(\sum_k a_{jk} b_{ki} \right)_{i,j}$$

$$= \left(\sum_k b_{ki} a_{jk} \right)_{i,j}$$

$A^T = (a^T_{ij})_{i,j}$ where $a^T_{ij} = a_{ji}$

swapping indicies

$$= \left(\sum_k b^T_{ik} a^T_{kj} \right)_{i,j}$$

$$= B^T A^T$$

■

(c)

$AA^T \neq A^T A$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\neq$

$$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

■

(d)

Note that if we have a sub-matrix and all others zero, the bigger matrix behaves like the smaller one.

i.e. Let $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$, then

$$\begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & 0 \end{pmatrix} \in \mathcal{M}_{m\times m}$$

$$= \begin{pmatrix} A & \cdots \\ \vdots & 0 \end{pmatrix}$$

(This is just notation)

Take $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$ then

$$\begin{pmatrix} A & \cdots \\ \vdots & 0 \end{pmatrix}\begin{pmatrix} B & \cdots \\ \vdots & 0 \end{pmatrix} = \begin{pmatrix} AB & \cdots \\ \vdots & 0 \end{pmatrix}$$

why?

$$(ab)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$= 0 \text{ if } \quad i \geq 2 \text{ or } j \geq 2$$

i.e. all the other entries other than A or B.

This means the result from (c) proves for $M_{m \times m}(\mathbb{R})$, just put our matrix from (c) in upper left. ■