Written HW 2 Solutions:

1a)

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be an arbitrary 2×2 matrix.

Let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$BA = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix}$$

Gives 4 eqs in unknowns $b_{11}, b_{12}, b_{21}, b_{22}$

$$\cancel{a_{11}b_{11}} + a_{12}b_{21} = \cancel{b_{11}a_{11}} + b_{12}a_{21}$$

$$\Rightarrow \quad a_{12}b_{21} - a_{21}b_{12} = 0$$

$$a_{11}b_{12} + a_{12}b_{22} = b_{11}a_{12} + b_{12}a_{22}$$

$$\Rightarrow \quad (a_{11} - a_{22})b_{12} + a_{12}b_{22} - a_{12}b_{11} = 0$$

$$a_{21}b_{11} + a_{22}b_{21} = b_{21}a_{11} + b_{22}a_{21}$$

$$\Rightarrow \quad a_{21}b_{11} + (a_{22} - a_{11})b_{21} - a_{21}b_{22}$$

$$a_{21}b_{12} + \cancel{a_{22}}b_{22} = b_{21}a_{12} + b_{22}\cancel{a_{22}}$$

$$\Rightarrow \quad a_{21}b_{12} - a_{12}b_{21} = 0$$

In matrix form

$$\underbrace{\begin{pmatrix} 0 & -a_{21} & a_{12} & 0 \\ -a_{12} & (a_{11}-a_{22}) & 0 & a_{12} \\ a_{21} & 0 & (a_{22}-a_{11}) & -a_{21} \\ 0 & a_{21} & -a_{12} & 0 \end{pmatrix}}_{\text{Gaussian Elimination}} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 0 & -a_{21} & a_{12} & 0 & 0 \\ -a_{12} & a_{11}-a_{22} & 0 & a_{12} & 0 \\ a_{21} & 0 & a_{22}-a_{11} & -a_{21} & 0 \\ 0 & a_{21} & -a_{12} & 0 & 0 \end{array}\right)$$

Row ops: $\sim$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \quad b_{11} = b_{22}, \quad b_{12} = b_{21} = 0$

So general form is

$$B = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \beta \in \mathbb{R}.$$

$\square$

1b) Recall from lec 5. that a subset is a v.s. if and only if it is closed under add & scalar mult.

Also recall $M_{m \times m}$ a v.s.

So we must show $Z \subset M_{m \times m}(\mathbb{R})$ is closed under + & scalar mult.

$+$: $A, B \in Z \Rightarrow AC = CA$ & $BC = CB$

$\forall \; C \in M_{m \times m}$.

$$\begin{aligned} (A+B)C &= AC + BC \\ &= CA + CB \\ &= C(A+B) \end{aligned}$$

So $(A+B) \in Z$

Mult: Let $\alpha \in \mathbb{R}$, $A \in Z$, then

$$\begin{aligned}(\alpha A)C &= \alpha(AC)\\ &= \alpha(CA)\\ &= C(\alpha A)\end{aligned}$$

So $\alpha A \in Z$.

By our theorem, Z is a v.s. $\blacksquare$

2a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$AA = \begin{pmatrix} \boxed{a^2 + bc} & \boxed{ab + bd} \\ \boxed{ac + cd} & \boxed{bc + d^2} \end{pmatrix}$$

Want : $= 0$

$\square$ If $b = 0$ or $c = 0$, the corresponding entry will be zero, so assume not. Then

$a + d = 0 \Rightarrow a = -d.$

$\square$ $a^2 + bc = 0 = d^2 + bc$

$\Rightarrow a^2 = -bc$

$\Rightarrow b = -\dfrac{a^2}{c}, \quad c \neq 0$

Two free parameters

or

So if $c \neq 0$

$$A = \begin{pmatrix} \alpha & -\frac{\alpha^2}{\beta} \\ \beta & -\alpha \end{pmatrix}, \quad \beta \neq 0$$

or: $$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ (corresponds to $\beta = 0$) $\square$

2b) No such matrix exists. This was a mistake. $\square$

HW Prob if we get far enough

2c) Let

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$ then

$$BA = \begin{pmatrix} a+c & \beta b + c & \alpha(a+b) + \beta c \\ d+f & \beta e + f & \alpha(d+e) + \beta f \\ g+i & \beta h + i & \alpha(g+h) + \beta i \end{pmatrix}$$

want

$= 0$

$$a + c = 0 \Rightarrow a = -c$$

$$d + f = 0 \Rightarrow d = -f$$

$$g + i = 0 \Rightarrow g = -i$$

$\beta b + c = 0 \Rightarrow b = c = 0$ so the eq is true $\forall \beta$

$$\Rightarrow a = 0$$

Similar to get

$$a = b = c = d = e = f = g = h = i = 0$$

so $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\square$

3) Recall from 21A that if f, g are continuous, so are $f+g$ & αf $\forall \alpha \in \mathbb{R}$.

Therefore C^0 is closed under $+$ & scalar mult.

We verify (1) - (8) from lecture 5

(1) Let $f, g, h \in C^0$, then

$$(f(x)+g(x))+h(x) = f(x)+g(x)+h(x)$$
$$= f(x)+(g(x)+h(x))$$
$$\forall x \in \mathbb{R}$$

(2) The function $z(x)=0 \in C^0$ and

$$z(x) + f(x) = f(x) \quad \forall f \in C^0$$
$$\forall x \in \mathbb{R}$$

(3) $\exists$ negatives, namely $-f$ given f

since $f(x) - f(x) = 0 \quad \forall x \in \mathbb{R}$

(4) $f(x)+g(x) = g(x)+f(x) \quad \forall x \in \mathbb{R}$

5. $\forall\ \alpha \in \mathbb{R},\ f, g \in C^0$

$$\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$$

$$\forall\ x \in \mathbb{R}$$

6. $\alpha, \beta \in \mathbb{R}$, then

$$(\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)$$

$$\forall\ x \in \mathbb{R}$$

7. $\forall\ \alpha, \beta \in \mathbb{R},\ f \in C^0$

$$(\alpha\beta) f(x) = \alpha(\beta f(x))$$

$$\forall\ x \in \mathbb{R}$$

8. $1 \in \mathbb{R}$ is the identity.

$\blacksquare$