HW 5 Solutions

1) Recall that $\det B = \det B^T$

and note $(A - \lambda I)^T = A^T - \lambda I$

then $\det(A - \lambda I) = \det(A^T - \lambda I)$

and A, A^T have same eigenvalues $\blacksquare$

2) Proving l.i. is sufficient since an m element l.i. set spans $\mathbb{R}^m$.

Suppose not, then $\exists$ eigenvectors x, y, z w/ eigenvalues $\xi, \lambda, \mu \in \mathbb{R}$, then $\exists$ non-zero α, β s.t.

$$x = \alpha y + \beta z$$

But

$$\begin{aligned} Ax &= \alpha A y + \beta A z \\ &= \alpha \lambda y + \beta \mu z \\ &= \xi x \quad \text{(since } x \text{ an eigenvector)} \end{aligned}$$

$$\Rightarrow \quad \xi x = \alpha \lambda y + \beta \mu z$$

$$x = \alpha y + \beta z$$

$$\Rightarrow \lambda = \xi = \mu$$

but we said distinct!

The contradiction shows L.I. $\blacksquare$

3) $A^k v = A^{k-1}(Av)$

$= A^{k-1}(\lambda v) = \lambda A^{k-1} v$

$= \lambda A^{k-2}(Av) = \lambda A^{k-2}(\lambda v)$

$= \lambda^2 A^{k-2} v$

$\vdots$

$= \lambda^j A^{k-j} v$

$\vdots$

$= \lambda^k v$ $\blacksquare$

4) $$\det(A - \lambda I) = \overbrace{\prod_{k=1}^{m} (\lambda_k - \lambda)}^{\text{Factored char. poly}}$$

set $\lambda = 0$

$\Rightarrow$

$$\det(A) = \prod_{k=1}^{m} \lambda_k$$

5) Singular iff $\det A = 0 = \prod_{k=1}^{m} \lambda_k$

$\iff \lambda_k = 0$ for some k

$\blacksquare$

(6a) $$e^{\begin{pmatrix}1&0\\0&1\end{pmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}e&0\\0&e\end{pmatrix}$$

$$e = \sum_{k=0}^{\infty}\frac{1}{k!}$$

$\square$

(6b)

$$A = \begin{pmatrix}1&-1\\1&-1\end{pmatrix}$$

$$A^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

So $$e^{\begin{pmatrix}1&-1\\1&-1\end{pmatrix}} = A^0 + A$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&-1\\1&-1\end{pmatrix}$$

$$= \begin{pmatrix}2&-1\\1&0\end{pmatrix}$$

$\square$

6c) Recall from HW 2 that nilpotent is either 0 or $\begin{pmatrix} \alpha & -\frac{\alpha}{\beta} \\ \beta & -\alpha \end{pmatrix}$, $\beta \neq 0$

for 0

$$e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^k$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^0$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & -\frac{\alpha}{\beta} \\ \beta & -\alpha \end{pmatrix}^2 = 0 \quad \text{(since nilpotent)}$$

$$e^{\begin{pmatrix} \alpha & -\frac{\alpha}{\beta} \\ \beta & -\alpha \end{pmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \alpha & -\frac{\alpha}{\beta} \\ \beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha & -\frac{\alpha}{\beta} \\ \beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1+\alpha & -\frac{\alpha}{\beta} \\ \beta & 1-\alpha \end{pmatrix}$$

$\square$