22A Lecture 2: Oct / 24 / 20

- Closure:

  → $\mathcal{M}_{m\times n}(\mathbb{C})$ under addition

Def: A set E is closed under the operation $\oplus$ if for every $a, b \in E$, $a \oplus b \in E$.

Ex $\mathbb{R}$ is closed under $+$

$a, b \in \mathbb{R} \quad a+b \in \mathbb{R}$

Thm: (Closure of $\mathcal{M}_{m\times n}(\mathbb{C})$ under addition & scalar multiplication)

Let $A, B \in \mathcal{M}_{m\times n}(\mathbb{C})$, then for every $\alpha \in \mathbb{C}$, $\alpha A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and

$A + B \in \mathcal{M}_{m\times n}(\mathbb{C})$.

Recall $\quad A = (a_{ij})_{m\times n} = \left(\cdots \right)$

$\alpha A = (\alpha a_{ij})_{m\times n}$, $\quad A + B = (a_{ij} + b_{ij})_{m\times n}$

↑ scalar multiplication

Pf: We have $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$ where $a_{ij}, b_{ij} \in \mathbb{C}$, $1 \le i \le m$, and $1 \le j \le n$. We know that $\mathbb{C}$ is closed under addition, and multiplication. Therefore αa_{ij} and $a_{ij} + b_{ij}$ are in $\mathbb{C}$ for all i, j. Therefore

$$\alpha A = (\alpha a_{ij})_{m \times n}$$ and

$$A + B = (a_{ij} + b_{ij})_{m \times n}$$ are

$m \times n$ matricies w/ complex entries. & therefore αA & $A+B$ are in $M_{m \times n}(\mathbb{C})$. $\blacksquare$

- Same proof w/ minor tweak gives closure for $M_{m \times n}(\mathbb{R})$.

$\alpha \in \mathbb{R}$

$\forall$ = for every

- A digression on fields

  Idea is to generalize $\mathbb{R}$ and $\mathbb{C}$.

  Mathematicians like generality

* Don't memorize, look up on test.

Def: A field is a set $\mathbb{F}$, together w/ two operations $\oplus, \otimes$ such that $\forall$ $a, b, c \in \mathbb{F}$

① : closure under $\oplus$: $a \oplus b \in \mathbb{F}$

② : commutivity : $a \oplus b = b \oplus a$

③ : associtivity : $a \oplus (b \oplus c)$

$$= (a \oplus b) \oplus c$$

④ : Zero : There is a $0 \in \mathbb{F}$ such that

$\forall$ $a \in \mathbb{F}$, $0 \oplus a = a$

⑤ : negatives : $\forall$ $a \in \mathbb{F}$ $\exists$ $(-a) \in \mathbb{F}$ such that $a \oplus (-a) = 0$

$\exists$ = there exist
$\forall$ = for every

SB: $a \oplus b = a+b$, $ab = a \otimes b$

(6): closure under $\otimes$: $\forall\, a, b \in \mathbb{F}$
$a \otimes b \in \mathbb{F}$

(7): associativity: $a \otimes (b \otimes c)$
$= (a \otimes b) \otimes c$

(8): commutative: $a \otimes b = b \otimes a$

unit element or identity

(9): one: $\exists\ 1 \in \mathbb{F}$ such that
$\forall\, a \in \mathbb{F}\quad a \otimes 1 = a$

(10): Recipricols: $\forall\, a \in \mathbb{F}$, $a \neq 0$,
$\exists\ a^{-1} \in \mathbb{F}$ such that $a \otimes a^{-1} = 1$.

(11): Distribution over addition:
$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

Ex $\mathbb{R}$ and $\mathbb{C}$ $\quad \oplus = +$, $\otimes = \times$

→ Field of two elements:

$\{0, 1\}$, $\quad 0 \oplus 0 = 0$
$1 \oplus 1 = 0$, $1 \oplus 1 = 0$

$0 \oplus 1 = 1 = 1 \oplus 0$

$1 \otimes 0 = 0$, $0 \otimes 0 = 0$

$\in$ = is a member of

$a \in E$

$1 \otimes 1 = 1$.

→ $C^\circ(\mathbb{R})$ is the space of continuous functions

almost a field

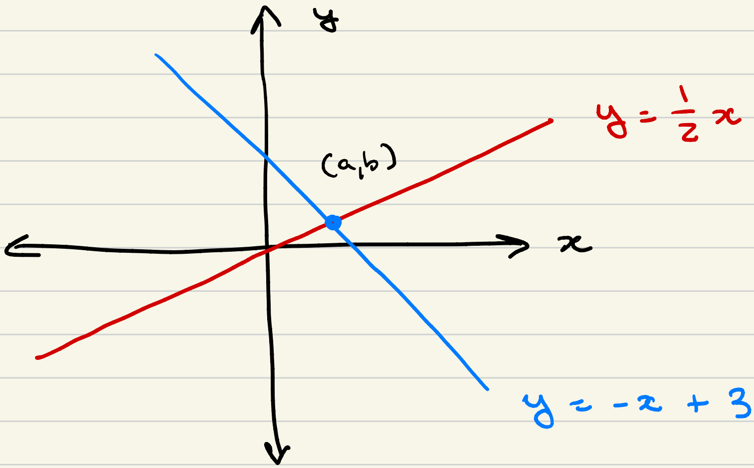
but $\{ f \in C^\circ(\mathbb{R}) \mid$ ethier $f > 0, f < 0\}$
and $\{f = 0\}$

$\mathbb{1} = 1$, $f = 0$ is zero $\quad f^{-1} = \frac{1}{f}$.

$\mathbb{1}$ = Identity element, one w/ two bars

- Solving linear Eqs

→ Some geometry



$$\begin{cases} \frac{1}{2}x - y = 0 \\ x + y = 3 \end{cases}$$

goal: find (a, b)

$$\frac{1}{2}x - \cancel{y} + x + \cancel{y} = 3$$

$$\frac{3}{2}x = 3 \Rightarrow \underline{x = 2}$$

$$x + y = 3$$

$$2 + y = 3 \Rightarrow y = 1$$

so $(a,b) = (2,1)$

$$\begin{pmatrix} 1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$\uparrow$ as of yet undefined

$\downarrow$ some manipulation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} x = a \\ y = b \end{cases}$$

- Equivalent systems of eqs.

A_1, A_2 are $m \times n$ matricies $M_{m\times n}(\mathbb{R})$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b_1 = \begin{pmatrix} b_1^1 \\ \vdots \\ b_n^1 \end{pmatrix}$$

$$b_2 = \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix}$$

We <u>define</u> equivalence as

$$A_1 x = b_1 \sim A_2 x = b_2$$

if the same x solves both eqs

Ex: $\begin{cases} x + y = 1 \\ -x + y = 3 \end{cases} \longleftrightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Solution: $x = -1, y = 2$

$$\begin{cases} 2x + 2y = 2 \\ -x + y = 3 \end{cases} \longleftrightarrow \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

solution: $x = -1,\ y = 2$

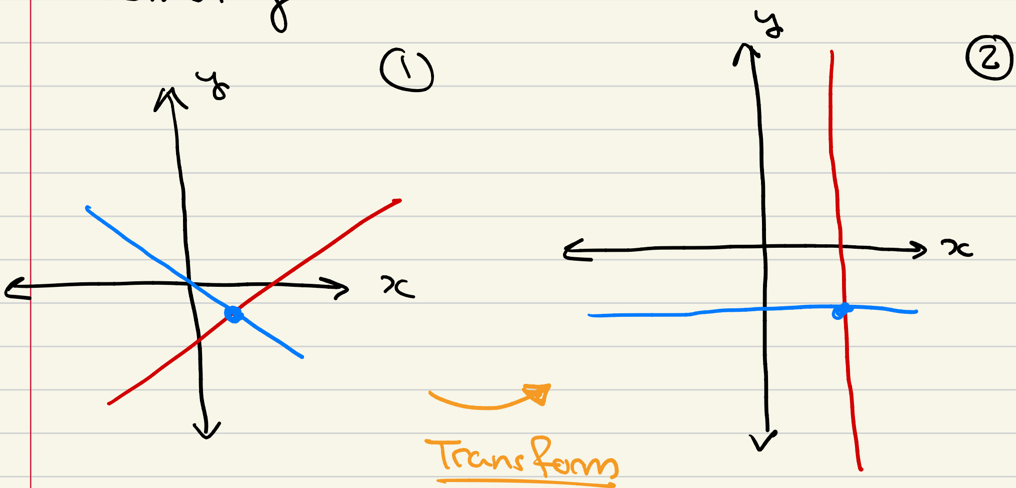
Says that multiplication of a line doesn't change solution

$$\begin{cases} x + 0y = -1 \\ 0x + y = 2 \end{cases}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

- Geometry



$x = a$
$y = b$

1 and 2 are equivalent, but 2 is much nicer

- Two tasks:

  (a) Find operations which preserve intersection point

  (b) Find a nice transformation

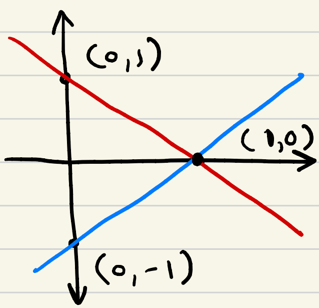
- The 3 row operations

  ① Interchange two rows

  ② Multiplying a row by a scalar

  ③ Adding two rows

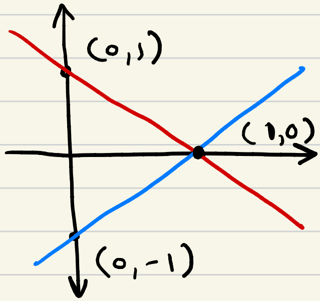
  ④ Adding a multiple of one row to another



$$\begin{cases} x + y = 1 \\ -x + y = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

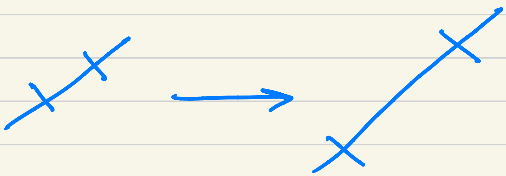
①: Interchanging rows



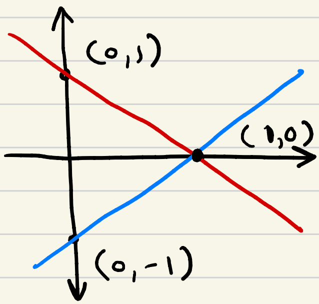
$$\begin{cases} -x + y = -1 \\ x + y = 1 \end{cases}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

②: Multiplication by a scalar (that isn't zero)



Multiply by 2



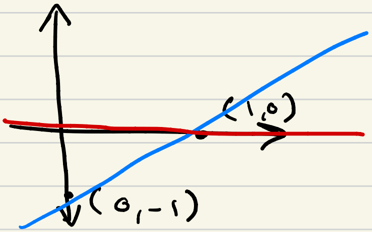
$$\begin{cases} -2x + 2y = -2 \\ x + y = 1 \end{cases}$$

$$\begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

* Most interesting

③ : Adding two rows

Add blue to red



$$\begin{cases} -x + y = -1 \\ \quad\;\; y = 0 \end{cases}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Given two <u>lines</u>

$$y = m_1 x + b_1 \quad , \quad y = m_2 x + b_2$$

Intersect at (α, β)

$$\beta = m_1 \alpha + b_1 = m_2 \alpha + b_2$$

Add lines

$$\begin{array}{rl} & y = m_1 x + b_1 \\ + & y = m_2 x + b_2 \\ \hline & 2y = (m_1 + m_2)x + (b_1 + b_2) \end{array}$$

$$\Rightarrow \quad y = \frac{1}{2}(m_1 + m_2)x + \frac{1}{2}(b_1 + b_2)$$

$$= \frac{1}{2}(m_1 x + b_1) + \frac{1}{2}(m_2 x + b_2)$$

$$x = \alpha \quad \downarrow$$

$$\frac{1}{2}(m_1 \alpha + b_1) + \frac{1}{2}(m_2 \alpha + b_2)$$

$$= \frac{1}{2}\beta + \frac{1}{2}\beta = \beta$$

$\square$

* Adding lines preserves intersection.

- Takeaway: solving linear equations amounts to figuring out what order to do the row operations.

(Reduced)

- Row Echelon Form

A matrix A is in REF if

(a) non-zero rows on top

(b) leading column number increases

(c) leading entries are 1

(d) A column w/ a leading 1 has only zero entries otherwise

$$\begin{pmatrix} 0 & 1 & x & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_{5\times 7}(\mathbb{R})$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Ex: $\begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_{3\times 5}(\mathbb{R})$

I = "identity matrix" = $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 1 \; 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Quiz solution:

(A)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(C)

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(D)

$$\begin{pmatrix} 0 & 1 & \overset{0}{\cancel{3}} & 0 & 0 \\ 0 & 0 & 1 & \boxed{2 \quad \cancel{3}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} 0$$

lead col: 2
3
5

increase

1b) $\oplus$ Associative means

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

$a \oplus (b \oplus c) =$

$=$

$=$

$=$

$=$

$=$

$=$

$= (a \oplus b) \oplus c$

Say $A, B, C \in M_{4 \times 4}(\mathbb{R})$