Lecture 3:

- Gaussian Elimination

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots \quad &= b_2 \\
\vdots \quad &\quad \vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_{mn} &= b_n
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
=
\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}
$$

$$
\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & & a_{mn} & b_n \end{array}\right) = (A \,\vdots\, b)
$$

- Augmented matrix

$$(A \,\vdots\, b) \longrightarrow (R \,\vdots\, c)$$

$A \to R$ by row operations

3 row ops.
1. multiply by a scalar
2. swap rows
3. add two rows

Ex:
$$2x + 3y = 1$$
$$x - y = 3$$

first pivot

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & -1 & 3 \end{array}\right) \underset{\sim}{R_1 = \tfrac{1}{2}R_1} \left(\begin{array}{cc|c} \boxed{1} & 3/2 & 1/2 \\ \boxed{1} & -1 & 3 \end{array}\right)$$

second pivot

$$\underset{\sim}{R_2 = R_2 - R_1} \left(\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ 0 & -5/2 & 5/2 \end{array}\right)$$

column is done

$$\underset{\sim}{R_2 = -\tfrac{2}{5}R_2} \left(\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ 0 & 1 & -1 \end{array}\right)$$

$$\underset{\sim}{R_1 = R_1 - \tfrac{3}{2}R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

Identity ; RREF

$$\begin{cases} x = 2 \\ y = -1 \end{cases}$$

Solution : $\boxed{x = 2,\ y = -1}$

$$2(2) + 3(-1) = 1$$
$$2 - (-1) = 3$$

✓

- Huerstic :
  1. Re order rows
  2. R_1 leading entry 1
  3. Make the column have only

$$\begin{matrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{matrix}$$

only one non-zero and non-zero entry is 1

  4. Repeat 2 & 3 until RREF

Ex:

$$\begin{aligned} 2x + 3y &= 1 \\ -x + 2y + 4z &= 2 \\ \boxed{x = 2} \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ -1 & 2 & 4 & 2 \\ \boxed{1 \quad 0 \quad 0} & & & 2 \end{array}\right) \underset{\sim}{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -1 & 2 & 4 & 2 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

looks nice

$$\underset{\sim}{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 4 & 4 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{R_3 = R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 4 & 4 \\ 0 & 3 & 0 & -3 \end{array}\right)$$

0 1 0

$$\underset{\sim}{R_2 \leftrightarrow \tfrac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 4 & 4 \end{array}\right)$$

$R_3 = R_3 - 2R_2$ $\sim$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & 6 \end{array}\right)$

$R_3 = \frac{1}{4}R_3$ $\sim$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \end{array}\right)$

$\Rightarrow$ solution $\boxed{x = 2,\ y = -1,\ z = 3/2}$

$$\begin{aligned} 2x + 3y + z &= 2 \\ x - y - z &= 1 \\ x + 6y - 3z &= -3 \end{aligned}$$

$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 1 & 6 & -3 & -3 \end{array}\right)$ $R_1 = \frac{1}{2}R_1$ $\sim$

$\left(\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 6 & -3 & -3 \end{array}\right)$

$$R_2 = R_2 - R_1$$
$$R_3 = R_3 - R_1$$
$$\sim \left(\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 1 \\ 0 & -5/2 & -3/2 & 0 \\ 0 & 9/2 & -7/2 & -4 \end{array}\right)$$

$$R_2 = -\frac{2}{5} R_2 \sim \left(\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 1 \\ 0 & 1 & 3/5 & 0 \\ 0 & 9/2 & -7/2 & -4 \end{array}\right)$$

$$R_1 = R_1 - \frac{3}{2} R_2 \sim \left(\begin{array}{ccc|c} 1 & 0 & 1/2 - \frac{9}{10} & 1 \\ 0 & 1 & 3/5 & 0 \\ 0 & 9/2 & -7/2 & -4 \end{array}\right)$$

$$R_3 = R_3 - \frac{9}{2} R_2 \sim \left(\begin{array}{ccc|c} 1 & 0 & 1/2 - \frac{9}{10} & 1 \\ 0 & 1 & 3/5 & 0 \\ 0 & 0 & -\frac{7}{2} - \frac{27}{10} & -4 \end{array}\right)$$

$$-\frac{7}{2} - \frac{27}{10}$$

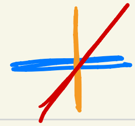
$$-\frac{35}{10} - \frac{27}{10} = -\frac{62}{10} = -\frac{31}{5}$$

$$R_3 = -\frac{5}{31} R_3 \sim \left(\begin{array}{ccc|c} 1 & 0 & -2/5 & 1 \\ 0 & 1 & 3/5 & 0 \\ 0 & 0 & 1 & \frac{20}{31} \end{array}\right)$$

$$R_1 = R_1 + \frac{2}{5} R_3$$
$$R_2 = R_2 - \frac{3}{5} R_3$$

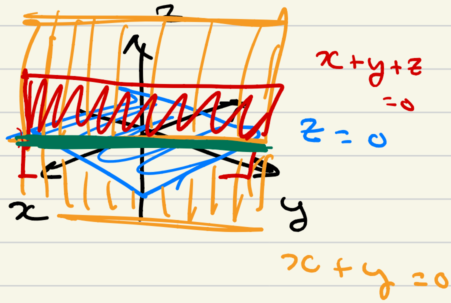
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 + \frac{2}{5} \frac{20}{31} \\ 0 & 1 & 0 & -\frac{3}{5} \times \frac{20}{31} \\ 0 & 0 & 1 & \frac{20}{31} \end{array}\right)$$

PW: gaussian L3:Q1



- Infinitely many solutions

$$\begin{aligned} z &= 0 \\ x + y &= 0 \\ x + y + z &= 0 \end{aligned}$$



$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \underset{\sim}{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 = R_2 - R_1} \quad \begin{array}{r} \\ -1 \rightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{aligned} R_2 &= -R_2 \\ R_1 &= R_1 - R_3 \\ R_3 &= R_3 - R_2 \end{aligned} \quad \sim \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

RREF

but <u>not</u> identity

$x+y=0$ & $z=0$

$\Downarrow$

$x=-y$ $\qquad\qquad \alpha\in\mathbb{R}$

$(\alpha, -\alpha)$

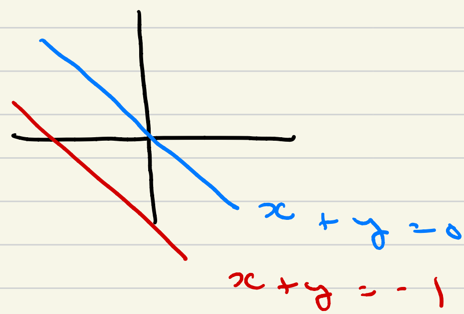
you give a real #, I give a solution

$$S=\{\underbrace{(\alpha,-\alpha)}_{\text{coords.}} \mid \alpha\in\mathbb{R}\}$$

such that

value is in $\mathbb{R}$

- No solutions



$$\left(\begin{array}{cc|c}1&1&0\\1&1&-1\end{array}\right) \overset{R_2=R_2-R_1}{\sim} \left(\begin{array}{cc|c}1&1&0\\0&0&-1\end{array}\right)$$

RREF

Reduced Row Ech. Form

$x + y = 0 \qquad 0x + 0y = \boxed{0 = -1}$

Red flag

tells us no solution

- Matrix multiplication

Goal: Make rigorous the correspondence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

column

$$\longleftrightarrow \quad \boxed{\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}}$$

row

column

$$\begin{pmatrix} \boxed{a \quad b} \\ \boxed{c \quad d} \end{pmatrix}\begin{pmatrix} \boxed{\begin{matrix} x \\ y \end{matrix}} \end{pmatrix} = \begin{array}{c} \begin{matrix} a & & b \\ x & & y \end{matrix} \\ \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \end{array}$$

$$= \boxed{\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

column vec.

$A = (a_{ij})_{m \times n}$ $B = (b_{ij})_{n \times p}$

- Definition:

$\mathbb{R}$ or $\mathbb{C}$

Given $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, $B \in \mathcal{M}_{n \times p}(\mathbb{F})$

We define $AB \in \mathcal{M}_{m \times p}(\mathbb{F})$ by

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{m \times p}$$

$$(ab)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Only defined when # of colums in first matrix = # of rows in second

must match

$$(m \times \boxed{n)(n} \times p)$$

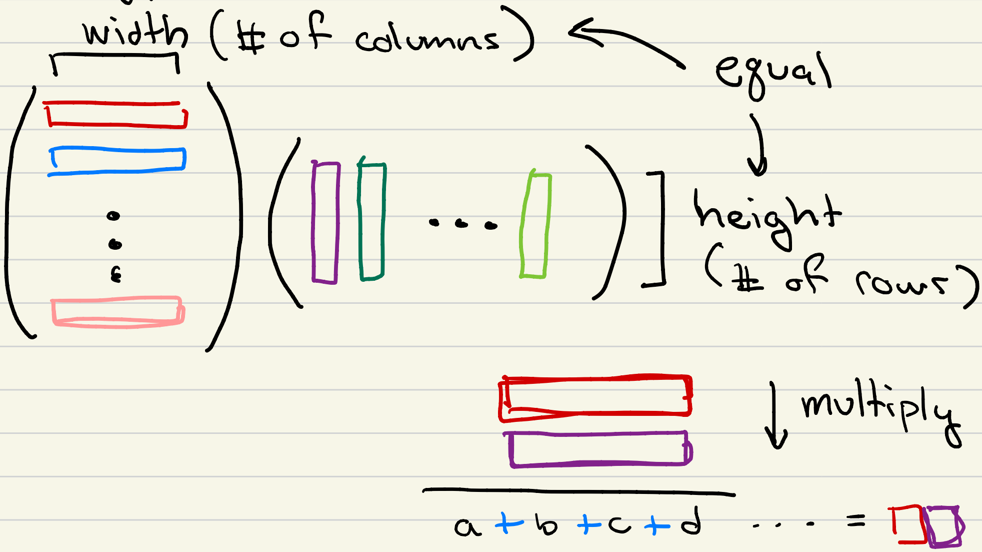
Note AB may be defined but BA may not.

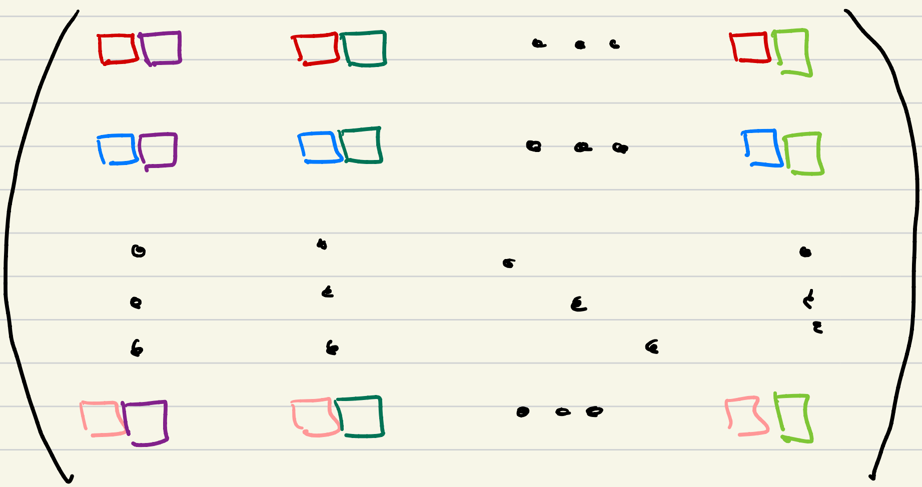
$$
\overset{m \times n}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}
\overset{n \times p}{\begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}}
$$

$$
= \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & c_{ij} & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix} \longleftarrow m \times p
$$

$$
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \begin{array}{l} 1 \le i \le m \\ 1 \le j \le p \end{array}
$$

- Visual





Ex: $\boxed{3 \times \boxed{2 \qquad 2} \times 3} \Rightarrow 3 \times 3$ out

$$\begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 6 & 3 & 2 \\ 2 & -1 & -1 \end{pmatrix}$$

$$\begin{array}{cc} 2 & 3 \\ 6 & 2 \\ \hline \multicolumn{2}{c}{12 + 6} \end{array} = 18 = \square\square$$

$$\begin{pmatrix} 18 & 2(3) + (3)(-1) & 2(2) + 3(-1) \\ 1(6) + 2(-2) & 3 + 2 & 2 + 2 \\ 24 + 2 & 12 - 1 & 8 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 3 & 1 \\ 2 & 5 & 4 \\ 26 & 11 & 7 \end{pmatrix}$$

$a, b, c, d \in \mathbb{R}$

Ex

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$2 \times \boxed{2 \quad 2} \times 2 \Rightarrow 2 \times 2$ out

$$\begin{pmatrix} a+2b & 2a+b \\ c+2d & 2c+d \end{pmatrix}$$

So now $\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2+2(-1) & 2(2)-1 \\ 4+2(6) & 2(4)+6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 \\ 16 & 14 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

$$\begin{array}{cc} 1 & 3 \\ 2 & 1 \\ \hline \end{array}$$

$$2 + 3 = 5$$

- Proofs:

Claim $(A+B)C = AC + BC$ when mult & addition are defined

Pf: $A, B \in M_{n \times m}(\mathbb{R})$, $C \in M_{m \times p}(\mathbb{R})$,

$A = (a_{ij})_{n \times m}$, $B = (b_{ij})_{n \times m}$

$C = (c_{ij})_{n \times p}$

We know

$$A + B = (\underbrace{a_{ij} + b_{ij}})_{n \times m}$$

$$\alpha_{ij} = a_{ij} + b_{ij}$$

$$\bar{\alpha} = A + B$$

We know $\bar{\alpha} C = \left(\sum_{k=1}^{m} \alpha_{ik} c_{kj} \right)_{n \times p}$

$= (A+B)C$

$$\underline{(A+B)C} = \left(\sum_{k=1}^{m} (a_{ik} + b_{ik}) c_{kj}\right)_{n\times p}$$

$\swarrow$ all $\mathbb{R}$

$$= \left(\sum_{k=1}^{m} a_{ik} c_{kj} + \sum_{k=1}^{m} b_{ik} c_{kj}\right)_{n\times p}$$

$$= \underbrace{\left(\sum_{k=1}^{m} a_{ik} c_{kj}\right)_{n\times p}}_{AC} + \underbrace{\left(\sum_{k=1}^{m} b_{ik} c_{kj}\right)_{n\times p}}_{BC}$$

$$= \underline{AC + BC} \qquad \square$$