Lecture 4: 06/29/20

Midterm: Fri July 10

through Mon July 6 lecture

- Finish matrix mult.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^{2} a_{ik} b_{kj}$$

$i = 1 \qquad j = 2 \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22}$

3×3

$A \quad B$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \cdots \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & \\ \vdots & \ddots \end{pmatrix}$$

L4 : Q1 : pw: more_mult

- Matrix transpose:

Given A, we call A^T the matrix transpose: "flip along diagonal"

Formally: $A = (a_{ij})_{\boxed{m \times n}}$

$$A^T = (a_{ji})_{\boxed{n \times m}}$$
$$= (a^T_{ij})_{n \times m}$$

Ex:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 7 \\ -1 & 2 \end{pmatrix}, \quad A^T = \begin{pmatrix} 2 & 4 & -1 \\ 3 & 7 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 4 & 0 \\ -1 & 2 & 6 \\ 4 & 3 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & -1 & 4 \\ 4 & 2 & 3 \\ 0 & 6 & 1 \end{pmatrix}$$

- L4: Q2, pw: transp

- Matrix Inverse:

  → We have matrix mult. Question, is there matrix division?

Def: $M_{m \times m}(\mathbb{R})$ are "square"

Real numbers

$ab = ac \Rightarrow b = c, \ a \neq 0$

(divide by a)

We can't do this w/ matricies

$AB \neq BA, \quad A, B \in M_{m \times m}$

for reals $a\cancel{b} = \cancel{b}c \Rightarrow a = c$

We wish to abstract the following notion:

reals $\frac{a}{a} = aa^{-1} = 1$

Note: if $A \in \mathcal{M}_{m \times n}(\mathbb{R})$ we have a L & R inverse

$A_L^{-1} A = I \qquad A A_R^{-1} = I, \qquad A_L^{-1} \neq A_R^{-1}$ in general

We want, given any $A \in \mathcal{M}_{m \times m}(\mathbb{R})$

find $A^{-1} \in \mathcal{M}_{m \times m}(\mathbb{R})$ such that

$$A A^{-1} = I$$

$$A^{-1} A = I \qquad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$$

Theoretically: We can compute a general formula for $A^{-1} \in \mathcal{M}_{m \times m}(\mathbb{R})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$ ← Want to find these

$$A A^{-1} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a\alpha + 0\beta + b\gamma + 0\delta = 1$

$$\begin{cases} a\alpha + b\gamma = 1 \\ a\beta + b\delta = 0 \\ c\alpha + d\gamma = 0 \\ c\beta + d\delta = 1 \end{cases}$$

four eqs.

four unknown

$\alpha, \beta, \gamma, \delta$ unknown

$$\sim \left(\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{array}\right)$$

$R_1 = \frac{1}{a} R_1$

$R_2 = \frac{1}{a} R_2$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{array}\right)$$

$R_3 = R_3 - cR_1$

$R_4 = R_4 - cR_2$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ 0 & 0 & \frac{ad-bc}{a} & 0 & -c/a \\ 0 & 0 & 0 & \frac{ad-bc}{a} & 1 \end{array}\right)$$

$$R_3 = \frac{a}{ad-bc} R_3$$

$$R_4 = \frac{a}{ad-bc} R_4$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ 0 & 0 & 1 & 0 & -\frac{c}{ad-bc} \\ 0 & 0 & 0 & 1 & \frac{a}{ad-bc} \end{array}\right)$$

$$R_1 = R_1 - \frac{b}{a} R_3$$

$$R_2 = R_2 - \frac{b}{a} R_4$$

$$\sim \left(\begin{array}{c|c} & \frac{1}{a} - \frac{b}{a}\left(-\frac{c}{ad-bc}\right) \\ I & -\frac{b}{a}\left(\frac{a}{ad-bc}\right) \\ & -\frac{c}{ad-bc} \\ & \frac{a}{ad-bc} \end{array}\right)$$

$$= \left(\begin{array}{c|c} & \frac{\cancel{a}d - \cancel{bc} + \cancel{bc}}{\cancel{a}(ad-bc)} \\ I & -\frac{b}{ad-bc} \\ & -\frac{c}{ad-bc} \\ & \frac{a}{ad-bc} \end{array}\right)$$

$$\alpha = \frac{d}{ad-bc}, \quad \beta = -\frac{b}{ad-bc}$$

$$\gamma = -\frac{c}{ad-bc} \qquad \delta = \frac{a}{ad-bc}$$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

invertible means you can take inv.

Note: $ad-bc \neq 0$

- A 2×2 matrix is invertible if and only if $ad-bc \neq 0$

- $ad-bc$ is called the <u>determinant</u> of A.

- We have been solving eqs of the form

$$Ax = b$$

if A is invertible, then the solution is

$$\boxed{x = A^{-1}b}$$

- Note that $x \neq bA^{-1}$, in fact bA^{-1} is <u>never</u> defined

- Consequence of non commutivity

$AB \neq BA$

$$Ax = b \quad \Rightarrow \quad A^{-1}Ax = A^{-1}b$$

A^{-1} A^{-1}

$$\underbrace{(A^{-1}A)}x = A^{-1}b$$

$$Ix = A^{-1}b$$

$$\Rightarrow \quad x = A^{-1}b$$

$\boxed{AXA^{-1} = BA^{-1}}$ (assuming this is allowed)

Not commutative

WE CANNOT SAY $AXA^{-1} = AA^{-1}X$

Important

1. keep track of which side you put the inverse

2. Matrix mult. does <u>not</u> commute.

- If A is square $A^{-1}A = I = AA^{-1}$

  if A not square

$$A_L^{-1}A = I = AA_R^{-1}$$

$$A_L^{-1} \neq A_R^{-1}$$

if $m \neq n$, these will be different sizes.

Ex:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

ad

$-bc$

First: check that it is invertible

$$ad - bc = 1\cdot 0 - (-1)(2)$$

$$= \underline{\underline{2}} \neq 0$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$$

Check

$$AA^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1}A = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Goal: Gaussian Elim. until left is I. (row ops)

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right)$$

↑ we want I here

$R_2 = R_2 + R_1$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right)$$

$R_2 = \frac{1}{2} R_2$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right)$$

$R_1 = R_1 - 2R_2$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right)$$

A^{-1}

$$A^{-1} = \begin{pmatrix} 0 & -1 \\ 1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$$

L4: Q3 pw: little_inv

Claim $(AB)^{-1} = B^{-1}A^{-1}$

Pf: $(AB)(AB)^{-1} = (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$ associtivity

$= AIA^{-1}$

$= AA^{-1}$

$= I$ ⟵ ☒

Claim: $(A^{-1})^{-1} = A$ Note $I^{-1} = I$

want to show

Pf: WTS $A^{-1}(A^{-1})^{-1} = I$

$I = A^{-1}A$

$$I = A^{-1} A$$

$$\Rightarrow \quad I^{-1} = (A^{-1} A)^{-1} = A^{-1} (A^{-1})^{-1}$$

$$I =$$

Ex:
$$\begin{pmatrix} 1 & 0 & 3 \\ 4 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

We can't check yet, don't know determ. for 3x3.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R_2 = R_2 - 4R_1$

$R_3 = R_3 - 2R_1$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & -11 & -4 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right)$$

$R_2 = \frac{1}{2} R_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{11}{2} & -2 & 1/2 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right)$$

$R_3 = R_3 + R_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{11}{2} & -2 & 1/2 & 0 \\ 0 & 0 & \boxed{-5 - \frac{11}{2}} & -4 & 1/2 & 1 \end{array}\right)$$

$$-\frac{21}{2}$$

$R_3 = -\frac{2}{21} R_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{11}{2} & -2 & 1/2 & 0 \\ 0 & 0 & 1 & \frac{8}{21} & -\frac{1}{21} & -\frac{2}{21} \end{array}\right)$$

$$\frac{21-11}{2(21)} = \frac{5}{21}$$

$R_1 = R_1 - 3R_3$

$R_2 = R_2 + \frac{11}{2}R_3$

$$\frac{-42+44}{21} = \frac{2}{21}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \boxed{-2+\frac{44}{21}} & \boxed{\frac{1}{2}-\frac{11}{2}\times\frac{1}{21}} & -\frac{11}{21} \\ 0 & 0 & 1 & 8/21 & -\frac{1}{21} & -\frac{2}{21} \end{array}\right)$$

A^{-1}

$$A^{-1} = \frac{1}{21}\begin{pmatrix} -3 & 3 & 6 \\ 2 & 5 & -11 \\ 8 & -1 & -2 \end{pmatrix}$$

pw: inv_2

$$-\frac{1}{7} = \underline{\underline{-3}} \times \frac{1}{21}$$

$$\begin{pmatrix} 3 & 6 & 4 \\ 1 & 9 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

1 $1/3$

$$\left(\begin{array}{ccc|ccc} 3 & 6 & 4 & 1 & 0 & 0 \\ 1 & 9 & 3 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \longleftrightarrow R_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 9 & 3 & 0 & 1 & 0 \\ 3 & 6 & 4 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

want zero

want zero

$R_2 = R_2 - 3R_1$

$R_3 = R_3 - 2R_1$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 9 & 3 & 0 & 1 & 0 \\ 0 & -21 & -5 & 1 & -3 & 0 \\ 0 & -17 & -2 & 0 & -2 & 1 \end{array}\right)$$

want to be 1

$R_2 = -\frac{1}{21} R_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 9 & 3 & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{21} & -\frac{1}{21} & \frac{3}{21} & 0 \\ 0 & -17 & -2 & 0 & -2 & 1 \end{array}\right)$$

want zero

want zero

we're done

$R_1 = R_1 - 9R_2$

$R_3 = R_3 + 17R_2$

$\sim$

$$\frac{18}{21} = \frac{6}{7} \qquad \frac{\frac{6}{7}}{3}\ ? \qquad \frac{9}{21} = \frac{3}{7} \qquad -\frac{6}{21} = -\frac{2}{7}$$

$$\frac{18}{21} = \frac{63-45}{21}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{3 - \frac{45}{21}} & \boxed{\frac{9}{21}} & \boxed{1 - \frac{27}{21}} & 0 \\ 0 & 1 & \frac{5}{21} & -\frac{1}{21} & \frac{3}{21} & 0 \\ 0 & 0 & \boxed{-2 + \frac{\overset{17\times 5}{85}}{21}} & -\frac{17}{21} & \boxed{-2 + \frac{51}{21}} & 1 \end{array}\right)$$

$$\frac{-42+85}{21} \quad \boxed{= \frac{43}{21}}$$

want $= 1$

$$\begin{array}{ccc} \frac{3}{7} & -\frac{2}{7} & 0 \\ -\frac{1}{21} & \frac{3}{21} & 0 \\ -\frac{17}{21} & \frac{3}{7} & 1 \end{array}$$

$$R_3 = \frac{21}{43} R_3$$

$$\sim$$

zero

zero

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{6/7} & 3/7 & -2/7 & 0 \\ 0 & 1 & \boxed{5/21} & -1/21 & 3/21 & 0 \\ 0 & 0 & 1 & -\frac{17}{43} & \frac{9}{43} & \frac{21}{43} \end{array}\right)$$

$$R_1 = R_1 - \frac{6}{7} R_3$$

$$R_2 = R_2 - \frac{5}{21} R_3$$

$$\sim$$

$$\left(\begin{array}{c|c} I & \underbrace{\begin{matrix} \frac{3}{7} + \frac{6}{7}\left(\frac{17}{43}\right) & -\frac{2}{7} - \frac{6}{7}\left(\frac{9}{43}\right) & -\frac{6}{7}\left(\frac{21}{43}\right) \\ -\frac{1}{21} + \frac{5}{21}\left(\frac{17}{43}\right) & \frac{3}{21} - \frac{5}{21}\left(\frac{9}{43}\right) & -\frac{5}{43} \\ -\frac{17}{43} & \frac{9}{43} & \frac{21}{43} \end{matrix}}_{A^{-1}} \end{array}\right)$$

$$\boxed{\frac{1}{43}\begin{pmatrix} \frac{135}{7} & -\frac{94}{7} & -18 \\ -\frac{38}{21} & \frac{124}{21} & -5 \\ -17 & 9 & 21 \end{pmatrix}}$$