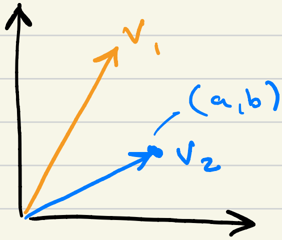
Lecture 5: 07/01/20

- Vectors (fancy way to think about $\mathbb{R}^n$)

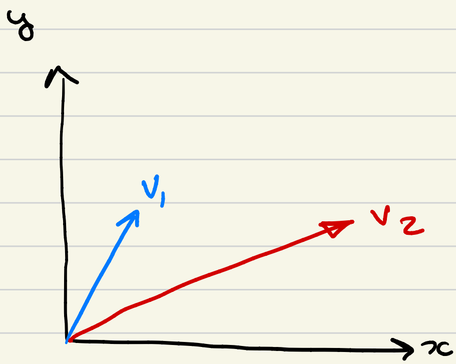


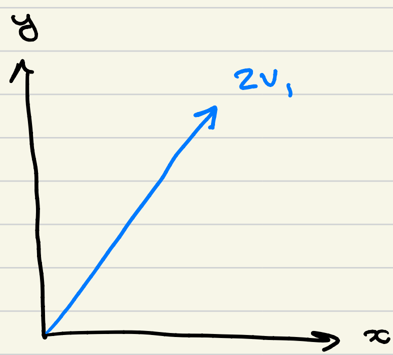
vector : magnitude & direction

We can represent as coordinates

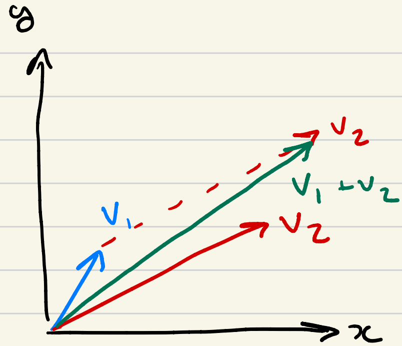
(a,b) length by Pyt. Thm. is $\sqrt{a^2+b^2}$ = magnitude

$$\frac{1}{\sqrt{a^2+b^2}}(a,b)$$





scalar mult.



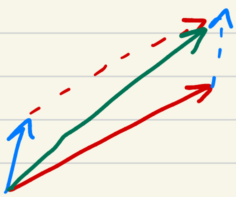
$v_1 + v_2$

$(a_1, b_1) + (a_2, b_2)$

$= (a_1 + a_2, b_1 + b_2)$

since + on $\mathbb{R}$ is commutative

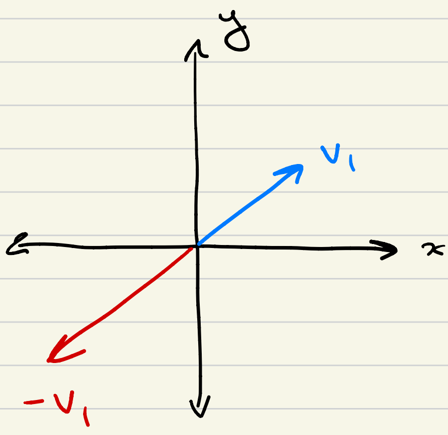
vector + is too



$v_1 + v_2 = v_2 + v_1$

$v_1 = (a_1, b_1)$

$\alpha v_1 = (\alpha a_1, \alpha b_1)$



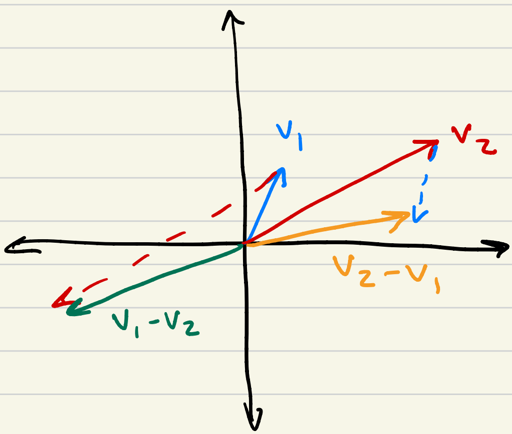
- Once we have addition and multiplication by -1, we get subtraction.

$$v_1 = (a_1, b_1), \quad v_2 = (a_2, b_2)$$

$$v_1 - v_2 = (a_1 - a_2, b_1 - b_2)$$

$$= v_1 + (-v_2)$$

$$v_1 - v_2 \neq v_2 - v_1$$



$$v_1 - v_2 = -(v_2 - v_1)$$

$v_1 = (a_1, a_2, \dots, a_m)$, $v_2 = (b_1, \dots b_m)$

$v_1, v_2 \in \mathbb{R}^m$, $v_1 - v_2 = -(v_2 - v_1)$

Pf: Generalize to higher dimensions

Recall that $\mathbb{R}^m \cong \mathcal{M}_{1\times n}(\mathbb{R})$

treat v_1, v_2 as row vectors

$$v_1 - v_2 = (a_{ij} - b_{ij})_{1\times n}$$

$$= (-(b_{ij} - a_{ij}))_{1\times n}$$

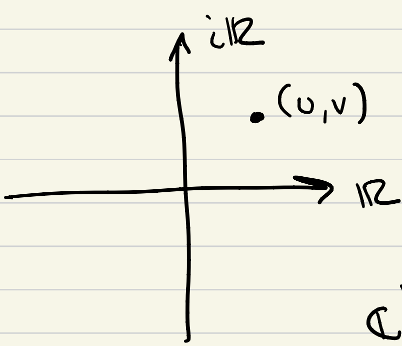
$$= -(b_{ij} - a_{ij})_{1\times n}$$

$$= -(v_2 - v_1)$$

$\boxed{\text{Ed}}$

Complex

$a = u + iv$, $i = \sqrt{-1}$



$\cong \mathbb{R}^2$

$\mathbb{C}^m \cong \mathbb{R}^{2m}$

$\forall$ = for all

$\exists$ = there exists

$\cdot\ni\cdot$ = such that s.t. $\boxed{\cdot\ni\cdot}$

- Big idea: Vector Spaces

Def: A vector space is a <u>nonempty</u> set V over a field $\mathbb{F}$ with two operations

- Addition : closed $\forall\ u, w \in V$

  $u + w \in V$

  When we say "over $\mathbb{F}$" we mean mult.

- Scalar Mult: $\forall\ \boxed{\alpha \in \mathbb{F}}\ u \in V$

  $\alpha u \in V$

Furthermore, V must satisfy the following:

1. Associtivity: $\forall\ u, w, z \in V$

$$(u + w) + z = u + (w + z)$$

2. zero: $\exists\ 0 \in V \cdot\ni\cdot\ \forall\ u \in V$

$$u + 0 = 0 + u = u$$

3. Negatives: $\forall\ u \in V,\ \exists\ (-u) \in V \cdot\ni\cdot$

$$u + (-u) = 0$$

④ Commutative: $\forall\ u, w \in V$

$$u + w = w + u$$

⑤ Scalar. Mult. Distributes over Addition

$\forall\ \alpha \in \mathbb{F},\ \forall\ u, w \in V$

$$\alpha(u + w) = \alpha u + \alpha w$$

⑥ $\forall\ \alpha, \beta \in \mathbb{F},\ \forall\ u \in V$

$$(\alpha + \beta)u = \alpha u + \beta u$$

⑦ $\forall\ \alpha, \beta \in \mathbb{F},\ \forall\ u \in V$

$$(\alpha\beta)u = \alpha(\beta u)$$

Question: does $(\alpha\beta)u = (\beta\alpha)u$? why or why not?

⑧ Identity: $\exists\ \mathbb{1} = 1 \in \mathbb{F}\ \ni\ \forall\ u \in V$

$$\mathbb{1}u = 1u = u$$

Think: "$\mathbb{R}^m$ over $\mathbb{R}$"

Vector space has four components

- Nonempty set V ← we call these vectors
- A field $\mathbb{F}$ ← we call these scalars
- An operation $+$
- An operation $\times$

Claim: The inverse element is unique.

Take away: We do not know what V is, $\mathbb{F}$ is, $+$ is, $\times$ is.

Pf: Let $u + a = 0$, $u + b = 0$

WTS: $a = b$

$$u + a = 0 = u + b$$

$$a + (u + a) = a + (u + b)$$

$$(a + u) + a = (a + u) + b \qquad \text{associtivity}$$

1

$0 + a = 0 + b$ by (2)

$a = b$ ∎

Inverse element in THIS context is $(-v)$ the negative.

pw: civ

- Ex. of vector spaces:

$\mathcal{M}_{m \times n}(\mathbb{R})$ over $\mathbb{R}$ is a vector space

$+$ is normal addition, mult is normal scalar matrix mult.

matrix zero

(2) $\bar{0} = (0)_{m \times n}$

$\forall \ A \in \mathcal{M}_{m \times n}(\mathbb{R})$

$$\bar{0} + A = (0 + a_{ij})_{m \times n} = (a_{ij} + 0)_{m \times n}$$

$$= A + \bar{0} = (a_{ij} + 0)_{m \times n}$$

$$= (a_{ij})_{m \times n} = A$$

- $\mathcal{R}([a,b])$: the space of Riemann integrable functions on $[a,b]$.

$$f \in \mathcal{R}([a,b]) \text{ if } \underbrace{\int_a^b f \, dx}_{\text{has to make sense}} < \infty$$

Defined via a limit of partial sums

$$\int_a^b f \, dx = \lim_{N\to\infty} \sum_{k=0}^{N-1} f\left(\frac{(b-a)k}{N} + a\right)$$

$f+g$ is as expected

and over $\mathbb{R}$ $\alpha \in \mathbb{R} \Rightarrow \alpha f$

- Closure under $+$, $f+g \in \mathcal{R}$

$$\underbrace{\int_a^b f+g \, dx}_{} = \underbrace{\int_a^b f \, dx + \int_a^b g \, dx}_{\text{makes sense}}$$

equality

$\Rightarrow$ $f+g \in \mathcal{R}$

- mult $\int_a^b \alpha f \, dx = \alpha \int_a^b f \, dx$

- Homogeneous System of Eqs

$Ax = b$ is homogeneous if $b = 0$

of the form $Ax = 0$

$A \in M_{m \times n}(\mathbb{R})$, $x \in M_{n \times 1}(\mathbb{R})$

Thing to note: if $x, y \in M_{n \times 1}(\mathbb{R})$
solve $Ax = 0$, that is $Ax = 0$ &
$Ay = 0$, then $x + y$ also solves

$$A(x+y) = Ax + Ay = 0 + 0 = 0$$

If $Ax = 0$, then $\forall\ \alpha \in \mathbb{R}$

$$A(\alpha x) = \alpha Ax = \alpha(Ax) = \alpha 0 = 0$$

So αx also solves our system.

→ Makes us think we have a vector space.

Indeed the space of all solutions to a homogeneous eq. is a v.s.

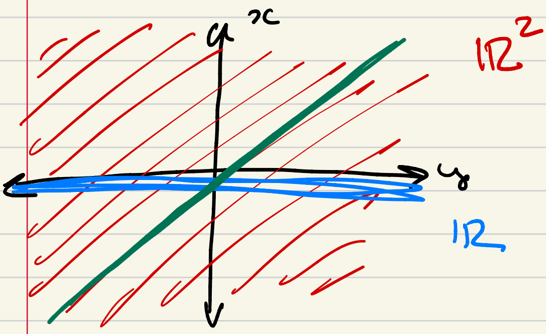
- Subspace

Def: given a v.s. V. A non-empty subset W, $(W \subset V)$ is a subspace if W is itself a vector space. ← over $\mathbb{R}$

Ex $\mathbb{R} \subset \mathbb{R}^2$ "$\mathbb{R}$ is a subspace of $\mathbb{R}^2$"

we know $\mathbb{R}^2$ is a vector space

$\mathbb{R}$ is a vector space.

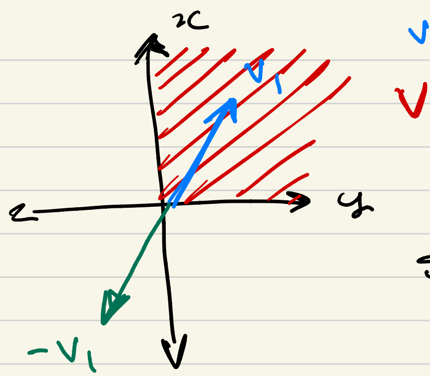


$(a,b) \in \mathbb{R}^2$

$(a,0) \in \mathbb{R}$

$(a,0),(b,0) \in \mathbb{R}$

$(a+b,0) \in \mathbb{R}$



$v_1 \in V$

$-v_1 \notin V$

so not closed under scalar mult
$\Rightarrow$ not a v.s.

Question 2: What def of $+, \times$ makes a quadrant into a v.s.?

Thm: Let V be a v.s. and let W be a non-empty subset of V. Then W is a subspace if and only if W is closed under addition & scalar mult.

Pf: W subspace $\Rightarrow$ closed under add. mult.

follows from def. of a subspace

W closed under add & mult $\Rightarrow$ subspace

(2) zero :

WTS: $\bar{0} \in W$, and $\forall\, w \in W$ $0 + w = w$

what we know

WWK : $\forall\, u \in V$, $u + \bar{0} = u$

W is closed under scalar mult.

so ... $\forall\, w \in W$, $0w = \bar{0}$, $\Rightarrow \bar{0} \in W$

Since $\forall\, u \in V$, $u + \bar{0} = u$

$w + \bar{0} = w \in W$.

(3) Negatives

WTS: $\forall\, w \in W$, $\exists\, (-w) \in W$ $\ni$

$w + (-w) = \bar{0}$

WWK : $\forall\, u \in V$, $\exists\, (-u) \in V$ $\ni$

$u + (-u) = \bar{0}$

because $w \in W \subset V$, $\exists\, (-w) \in V$ $\ni$

$w + (-w) = \bar{0}$

We also know $(-1)w = (-w)$

and because W closed under mult.

$-w \in W$. $\blacksquare$

pw: july-4

Quiz 2:

$$\underset{A}{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} \oplus \underset{B}{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} = \begin{pmatrix} a_1 + b_2 \\ a_2 + b_1 \end{pmatrix}$$

Associtive

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

let $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$A \oplus B = \begin{pmatrix} a_1 + b_2 \\ a_2 + b_1 \end{pmatrix}$$

$$(A \oplus B) \oplus C = \boxed{\begin{pmatrix} a_1 + b_2 + c_2 \\ a_2 + b_1 + c_1 \end{pmatrix}}$$

$$(B \oplus C) = \begin{pmatrix} b_1 + c_2 \\ \boxed{b_2 + c_1} \end{pmatrix}$$

#

$$A \oplus (B \oplus C) = \boxed{\begin{pmatrix} a_1 + b_2 + c_1 \\ a_2 + b_1 + c_2 \end{pmatrix}}$$

$$\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \vdots \\ \cdot & \cdot & \cdot & \vdots \\ 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \underline{\underline{0 = 1}}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & k & 2 \end{array}\right) \underset{\sim}{R_2 = R_2 + R_1} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2+k & 5 \end{array}\right)$$

want $k \ni .$

$2 + k = 0$

$k = -2$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 5 \end{array}\right) \Rightarrow \text{no sol.}$$

$$\left(\begin{array}{ccc|c} 1 & a & b & \\ 0 & 1 & c & \\ 0 & 0 & \underbrace{k + a}_{0} & \end{array}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\boxed{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

determinant

$A \in M_{m \times m}(\mathbb{R})$

det A.

some #

If det $A = 0$, then there is no solution

Rule: tells how to get # for $A \in M_{3\times3}(\mathbb{R})$

$k + q = \det A = 0$