Lecture 6:

- Vectors aren't always "vectors"

- Linear combinations

Idea is a sum of rescaled vectors

$\exists$ = "there exist"

Def: Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of vectors in a v.s. V. $(X \subset V)$

We call $w \in V$ a linear combination of X if and only if $\exists$ $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ "scalars"

s.t. (such that)

$$w = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

$$= \sum_{k=1}^{n} \alpha_k x_k$$

$\boxed{(\mathbb{R}^3, \mathbb{R}, +, \times)}$

$(V, \mathbb{F}, +, \times)$

Ex: $X = \{\overset{x_1}{(1,0,1)}, \overset{x_2}{(0,1,1)}\}$, $V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$

$$\underbrace{2}_{\alpha_1}\underbrace{(1,0,1)}_{x_1} + \underbrace{\overset{4}{3}}_{\alpha_2}\underbrace{(0,1,1)}_{x_2}$$

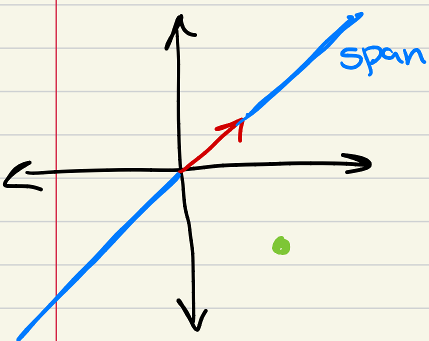
$$= (\overset{2}{2}, \overset{4}{3}, \overset{6}{5})$$

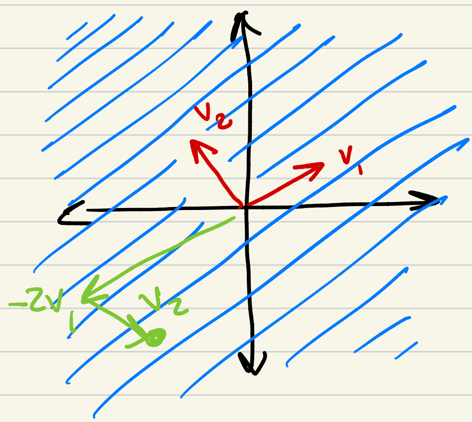
LG: Q1 pw: lin_com

- Span

Def: Given a v.s. V & a set (family) of vectors $X \subset V$, $X = \{x_1, \dots, x_n\}$, we denote by $\langle X \rangle$ or $\text{span}\{X\}$ the "span" of X, which is the set of <u>all</u> linear combinations of X.

Intuition: $\mathbb{R}^2$

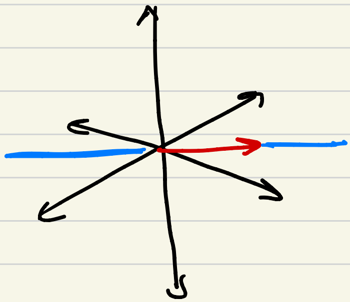


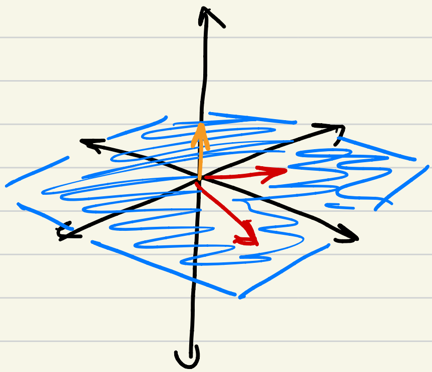


$-2v_1 + v_2$

$\alpha_1 = -2, \ \alpha_2 = 1$

- in $\mathbb{R}^3$





- Notation, if there are two sets X, Y, they use $\langle X, Y \rangle$ to mean the span of $X \cup Y$ = all elements of X and all elements of Y.

Thm: Let V be a v.s., $X \subset V$ a family of vectors. Then $\langle X \rangle \subset V$, and $\langle X \rangle$ is a subspace.

scalar mult w/ $\mathbb{F}$

Pf: Let $x \in \langle X \rangle$, then $x = \sum_{k=1}^{n} \alpha_k x_k$

vector $+$ or $\oplus$

where each $x_k \in X$. Since V is closed under $+$ and $\times$, $x \in V$

$\Rightarrow \langle X \rangle \subset V$. ✓

Recall that if a subset of a v.s. is closed under $+$ & $\times$, then that subset is a subspace.

$\oplus$: let $x, y \in \langle X \rangle$, then we can write

$$x = \sum_k \alpha_k x_k, \quad y = \sum_k \beta_k x_k$$

$$x + y = \left(\sum_k \alpha_k x_k \right) + \left(\sum_k \beta_k x_k \right)$$

$$= \sum_k \left(\alpha_k x_k + \beta_k x_k \right) \qquad 1$$

$$= \sum_k \underbrace{(\alpha_k + \beta_k)}_{\gamma_k} x_k \qquad 5$$

$$= \sum_k \gamma_k x_k \implies x + y \in \langle X \rangle$$

X: let x be as above, $\beta \in \mathbb{F}$

$$\beta x = \beta \left(\sum_k \alpha_k x_k \right)$$

$$= \sum_k \beta(\alpha_k x_k) \qquad 5$$

$$= \sum_k \underbrace{(\beta \alpha_k)}_{\gamma_k} x_k \qquad 7$$

$$= \sum_k \gamma_k x_k \implies \beta x \in \langle X \rangle$$

$\blacksquare$

L6Q2 pw: span

- Linear Independence & dependence

  Main idea: If given some vectors if any of those vectors are a linear combination of the others, then the set is linearly dependent.

Ex: $X = \{(0,1,0), (0,0,1), (0,1,1)\}$

X is L.D. since

$$(0,1,1) = (0,1,0) + (0,0,1)$$

$$\text{span } X = \langle X \rangle = \text{span}\{(0,1,0), (0,0,1)\}$$

implicitly a v.s. $\underline{\underline{V}}$ $\rightarrow X \subset V$

Def: Given a family $X = \{x_1, \ldots, x_n\}$ of vectors, we say X is L.D. if

$\exists$ scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ $\ni$

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \sum_{k=1}^{n} \alpha_k x_k$$

$$= 0$$

$\underline{\underline{\text{and}}}$ at least one $\alpha_i \neq 0$.

X is L.I. if X is not L.D.

Another way to think of L.I.

X is L.I. if

$$\sum_{k=1}^{n} \alpha_k x_n = 0$$

only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

$\{(1,0,0),(0,1,0),(0,0,1),(0,1,1)\}$

Ex: $\{\underset{e^1}{(1,0,0)}, \underset{e^2}{(0,1,0)}, \underset{e^3}{(0,0,1)}\} = E$

I claim $\underline{E \text{ spans } \mathbb{R}^3}$, and E is L.I.

$\langle E \rangle = \mathbb{R}^3$

linear combo: $a,b,c \in \mathbb{R}$

$$ae^1 + be^2 + ce^3 = (a,b,c)$$

$$\Rightarrow \langle E \rangle = \mathbb{R}^3$$

$$(a,b,c) = 0 = (0,0,0)$$

$$\Rightarrow a = 0 = b = c \Rightarrow \text{L.I.}$$

Ex: $X = \{\underset{x_1}{(1,0,0)}, \underset{x_2}{(0,1,0)}, \underset{x_3}{(1,1,0)}\}$

Claim: X is L.D. (not L.I.), do not span $\mathbb{R}^3$

$a,b,c \in \mathbb{R}$

$$ax_1 + bx_2 + cx_3 = (a+c, b+c, 0)$$

Doesn't span, since $(0,0,1) \notin \langle x \rangle$

$= 0 = (0,0,0)$

$a + c = 0 \Rightarrow a = -c$

$b + c = 0 \qquad b - a = 0$

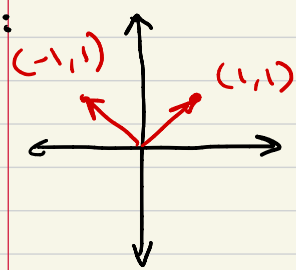
$0 = 0 \qquad b = a$

choose $a = 1,\ b = 1,\ c = -1$

$a x_1 + b x_2 + c x_3 = 0$

but $a, b, c \neq 0 \Rightarrow$ L. D.

Ex:



$a(-1,1) + b(1,1)$

$(-a+b,\ a+b)$

$= (0,0)$

$b = a$

$b = -a \Rightarrow \underline{\underline{a = -a}}$ $\in \mathbb{R}$

$\Rightarrow a = 0$

$\Rightarrow \underline{\underline{b = 0}}$

so L. I.

x_1, x_2, x_3

assume $a, b, c \neq 0$

$$c x_3 = a x_1 + b x_2$$

$$\iff \quad a x_1 + b x_2 - c x_3 = 0$$

- Brute Force Method:

Given $x_1, x_2, \dots, x_m \in \mathbb{R}^n \cong \mathcal{M}_{n \times 1}(\mathbb{R})$

$\boxed{A\alpha = 0}$ if only trivial sol ($\alpha = 0$) then l.i.

$$A = \begin{pmatrix} x_1 & x_2 & \dots & x_m \end{pmatrix} \in \mathcal{M}_{n \times m}(\mathbb{R})$$

$$= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & & & \vdots \\ \vdots & & & \\ & & \cdots & \end{pmatrix}$$

$$\alpha \in \mathcal{M}_{m \times 1}(\mathbb{R}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\{(1,0,0),\ (0,1,0),\ (1,1,0)\}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ax = 0$$
$$x = A^{-1}0$$
$$= 0$$

$$\left(\begin{array}{ccccc} 1 & 0 & 1 & 1 & 0 \\ 0 & \boxed{1 \quad 1} & & 1 & 0 \\ & & & 1 & \\ \underbrace{0 \quad 0 \quad 0}_{\text{RREF}} & & & & 0 \\ & & & 1 & \end{array}\right)$$

$y + z = 0$

non-trivial solution

$\downarrow$

$(1, 1, -1)$

$$x = -z$$
$$y = -z$$
$$\Rightarrow x = y$$

choose $x = 1$

Trivial solution $= (0,0,0)$

$$X = \{(1,1,0),\ (-1,0,1),\ (2,1,-1)\}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$R_2 \longleftrightarrow R_3 \quad \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$R_3 = R_3 - R_1 \quad \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$R_3 = R_3 - R_2 \quad \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Not L.I.   not equiv $\sim$ to I

$$x_1 = (x_{11}, x_{21}, x_{31})$$

$$\vdots$$

$$x_3 = (x_{13}, x_{23}, x_{33})$$

$$ax_1 + bx_2 + cx_3$$

$$= 0$$

$$ax_{11} + bx_{12} + cx_{13} = 0$$

$$ax_{21} + bx_{22} + cx_{23} = 0$$

$$ax_{31} + bx_{32} + cx_{33} = 0$$

$$\begin{pmatrix} x_{11} & & \\ & \ddots & \\ & & \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

LQ63 pw: ind

Def: A basis of a vector space is a l.i. spanning set.

So X is a basis of V if and only if

① X is l.i.

② $\langle X \rangle = V$ ← spanning set

X a spanning set if $\langle X \rangle = V$

END OF CONTENT FOR MT

Quiz 1:

$$\overset{x_1}{\begin{pmatrix} 1 & \boxed{0} & 1 \end{pmatrix}} \qquad \overset{x_2}{\begin{pmatrix} 0 & \boxed{0} & 1 \end{pmatrix}}$$

any linear combo will have 2nd component zero

$$\begin{pmatrix} 6 & 0 & -6 \end{pmatrix}$$

$$= 6x_1 - 12x_2$$

Quiz 2: span $\mathbb{R}^2$

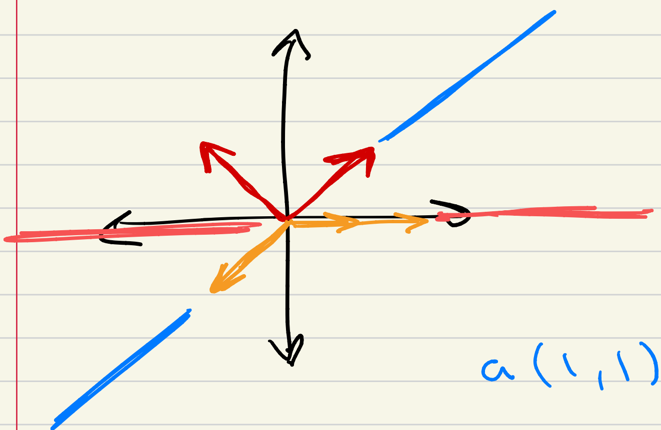
$(1,0)$ & $(2,0)$ ← $2(1,0)$ so L.D.

$\boxed{(1,1) \quad \& \quad (-1,1)}$ ← by process of elim

$(1,1)$ & $(-1,-1)$ ←

$(-1,-1) = -(1,1)$

so L.D.



$a(1,1) + b(-1,1)$

$(a-b,\ a+b) = 0, 0$

$a - b = c$

$a + b = d$

$a = b$

$a = -b$

$\Rightarrow b = -b$

$\Rightarrow b = 0$

$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\Rightarrow a = 0$

so L.I.

$$\sim \left(\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{array}\right)$$

L.I.

$a(1,0) = (a,0)$

Quiz 3:

(1) $(1,0)$ & $(1,-1)$ L.I.

-1

(2) $(1,0,1),(0,1,1)$ & $(1,-1,0)$

$$(1,-1,0) = (1,0,1)-(0,1,1)$$

so L.D.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$R_3 = R_3 - R_1$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{L.D.}$$

$x = -z$

$y = z$