Lecture 7: 07/08/20

Def: Let V be a v.s., X a family of vectors, $X \subset V$. X is a basis of V if and only if

1. $\langle X \rangle = V$
2. X is linearly indepent (L.I.)

"easy & illuminating consequence" + "deep"

Consq: Let V be a v.s. w/ basis

$$X = \{x_1, \ldots, x_n\}$$

then $\forall\, v \in V$, $\exists\, \alpha_1, \ldots, \alpha_n$ s.t.

$$v = \underbrace{\alpha_1 x_1 + \cdots + \alpha_n x_n = \sum_{k=1}^{n} \alpha_k x_k}_{\text{"Representation of } v \text{ under } X\text{"}}$$

"Representation of v under X".

both bases

Ex: $X = \{ \overset{x_1}{(1,0)}, \overset{x_2}{(0,1)} \} \subset \mathbb{R}^2$

$Y = \{ \overset{y_1}{(1,1)}, \overset{y_2}{(1,-1)} \} \subset \mathbb{R}^2$

$(a,b) = ax_1 + bx_2$ Rep. under X

Challenge: derive this $\left[\; = \frac{a+b}{2} y_1 + \frac{a-b}{2} y_2 \right.$ Rep. under Y

Key take away is that we can write vectors as linear combinations.

L7Q1 : pw: base

V is a v.s.

Thm: Given any set $S \subset V$, where $\langle S \rangle = V$, we can find a subset of S which is a basis of V.

Pf: Sketch: Remove vectors from S, until S is L.I.

Notation $A \subset B$ means $\forall\, a \in A,\ a \in B$

$A = B \Rightarrow A \subset B$ & $B \subset A$

- Dimension of a v.s.

Def: V is finite dimensional if and only if there is a finite family of vectors spanning V.

Thm: Let V be a v.s. and be finite dimensional (f.d.), then

1. There is a finite basis for V
2. All bases of V are the same size.

Pf in S-B
1. is a consequence of the Theorem above.
2. basically: given two bases X & Y, we take advantage of the fact that

$$X \subset \langle Y \rangle, \quad Y \subset \langle X \rangle$$

$|A| = \#$ of elements in A

$|X| = n, \ |Y| = m$

$X \subset \langle Y \rangle \Rightarrow n \leq m$, since otherwise X wouldn't be a base

$Y \subset \langle X \rangle \Rightarrow m \leq n$

$$\Rightarrow m = n$$

$$|X| = |Y|$$

$\blacksquare$

Def: The dimension of a v.s. V, denoted $\dim V$, is the size of any of its bases.

$\langle\{e^1, e^2, e^3\}\rangle = \mathbb{R}^3 \longrightarrow$ base b.c. $(a,b,c) = ae^1 + be^2 + ce^3$

L.I. $\Rightarrow$ base

Ex: $\mathbb{R}^3$ has $\dim \mathbb{R}^3 = 3$, since

$$\{\underset{e^1}{(1,0,0)}, \underset{e^2}{(0,1,0)}, \underset{e^3}{(0,0,1)}\}$$

is a basis of $\mathbb{R}^3$

Ex: $P_n[x]$: space of n-th degree polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 x$$

$\dim P_n[x] = n+1$

since

$\{1, x, x^2, \cdots, x^n\}$ is a basis.

Prop: Let V be a v.s. w/ $\dim V = n > 0$

1. Every L.I. family of vectors has <u>at most</u> n members.

2. Every family spanning V has <u>at least</u> n members.

- pf in book

→ Gives a nice way to check L.I.

Ex $\{(1,0,0),(1,1,0),(1,0,1),$
$(0,1,1)\} \subset \mathbb{R}^3$

Not L.I. bc 4 vectors, $4 \geq 3 = \dim \mathbb{R}^3$

So not L.I.

- L7Q2 pw: dim
- L7Q3 pw: dim 2

V a v.s.

Prop: Let $W \subset V$ be a subspace, then

$$\dim W \le \dim V.$$

If W a subspace, then $\dim W \le \dim V$

Proof Technique, contradiction:

if P, then Q

P	Q	$\underline{P \Rightarrow Q}$
0	0	1
0	1	1
1	0	0
1	1	1

same
$\longleftrightarrow$

P	Q	$\underline{\neg Q \Rightarrow \neg P}$
0	0	1
0	1	1
1	0	0
1	1	1

proof by cont. say $\neg Q \Rightarrow \neg P$

not

WTP: W a subspace $\Rightarrow \dim W \le \dim V$

same as $\Updownarrow$

$\dim W > \dim V \Rightarrow W$ not a subspace

Given $W \subset V$

Given W a subspace $\Rightarrow$ W is a v.s.

Pf: Suppose $\dim V < \dim W$, since W a v.s. $\exists\ X \subset W \ni |X| = \dim W$

$\&\ \langle X \rangle = W \rightarrow X$ is a base of W

X is L.I.

but $W \subset V \Rightarrow \langle X \rangle \subset V$

$\Rightarrow |X| \leq \dim V$

$|X| \begin{cases} \dim W \leq \dim V \\ \dim V < \dim W \end{cases}$ $\leftarrow \neg Q$

Contradiction

$\Rightarrow \dim W \leq \dim V$ Q $\blacksquare$

Take away: Gives a way to rule out W being a subspace.

- Four Fundamental Subspaces

Given a matrix $A \in M_{m \times n}(\mathbb{R})$

1. Row space $C(A^T)$
2. Column space $C(A)$
3. Nullspace $N(A)$
4. Left nullspace $N(A^T)$

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3. Nullspace: $N(A)$ consists of the solutions to $Ax = 0$

$$N(A) \subset \mathbb{R}^n$$

$$Ax = 0 \sim \begin{cases} w + y = 0 \\ \dots \end{cases}$$

Ex:

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{RREF}} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\sim$

$$\begin{cases} w + y = 0 \\ x + 2y = 0 \\ z = 0 \end{cases}$$

$$\boxed{\begin{aligned} w &= -y \\ x &= -2y \end{aligned}}$$

$-2 = -2(1)$

if we choose w, x, or y parameter

$(-1, -2, 1, 0)$ solves $Ax = 0$

$$A(\alpha x) = 0$$

$$= \alpha \underbrace{(Ax)}_{0} = 0$$

$$\langle \{(-2, 1, -1, 0)\} \rangle = N(A)$$

1D nullspace

$|N(A)| = \infty$

$\neq \dim N(A)$

$$\overset{3\times 4}{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \overset{4\times 1}{\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}} = \overset{3\times 1}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

w or z

$w + z = 0 \qquad w = -z$ Two

$x + y = 0 \qquad \boxed{x = -y}$ parameters

x or y

$(1, 0, 0, -1), \ (0, 1, -1, 0)$

$\langle \{(1, 0, 0, -1), (0, 1, -1, 0)\} \rangle$

$= N(A)$

Choose $(1, 1, -1, -1)$ Not base

$\langle \{(1, 1, -1, -1)\} \rangle \neq N(A)$

$(1, 0, 0, -1) \neq \alpha (1, 1, -1, -1)$

L7 Q4 pw: test

$$S = \{ \underbrace{(x_1, x_2, x_3)}_{\neq \mathbb{R}^3} \mid \underbrace{2x_1 + 3x_2 - 5x_3 = 0}\}$$

$(1,1,0) \in \mathbb{R}^3 \quad 2+3 \neq 0$

Q1: is $(1,0,0), (1,1,0), (0,1,1)$ a basis for $\mathbb{R}^3$

α

0

yes: L.I.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\underline{\underline{3}}$

$R_1 = R_1 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 = R_1 + R_3$

$R_2 = R_2 - R_3$

$$\sim \begin{pmatrix} I \end{pmatrix} \checkmark$$

$\rightarrow$ this means <u>base</u>

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{not a base}$$

Q2: What is the dimension of the subspace of $M_{2\times 2}$ spanned by

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

base

$\dim \langle S \rangle \leq 2$

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$\alpha = 0$

So L.I. $\quad 2 \leq \dim \langle S \rangle$

Given a L.I. set $\quad \langle S \rangle \subset V \quad \langle S \rangle \subset \langle S \rangle \rightarrow |S| \leq \dim V$

$\Rightarrow \dim \langle S \rangle = 2 \qquad |S| \leq \dim \langle S \rangle$

2

$\langle \{ (1,0,0,0), (0,1,0,0), (0,0,1,0) \} \rangle$

3 dimensional subspace of $\mathbb{R}^4$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = S$$

$|S| = 3$ $\quad V = \langle S \rangle$

$\langle S \rangle = \langle S \rangle$ so S spans $\langle S \rangle$

$\Rightarrow$

$\langle S \rangle = V \Rightarrow |S| \geq \dim V$

$|S| = 3 \geq \dim \langle S \rangle \leftarrow$

$3 =$

L.I. $\quad |S| \leq \dim V = \dim \langle S \rangle$

$\Rightarrow \quad 3 \leq \dim \langle S \rangle \leftarrow$

$3 \leq \dim \langle S \rangle \leq 3$

$\Rightarrow \quad \dim \langle S \rangle = 3.$

Q3: $\{ \overset{e^1}{(0,1,0,1)}, \overset{e^2}{(1,0,1,0)}, \overset{e^3}{(1,0,0,0)} \} \subset \mathbb{R}^4$

1 1 1

$\gamma(1,0,0,0) = (\beta,\alpha,\beta,\alpha)$

$\beta = 1$, doesn't work

$\underline{\alpha,\beta,\gamma = 0}$ $\Rightarrow$ L.I.

Dim 2

$\{(1,0),(0,1),(1,1)\} \subset M_{1\times 2}(\mathbb{R}) \cong \mathbb{R}^2$

$3 \geq 2 \Rightarrow$ L.D.

$\alpha e^1 + \beta e^2 + \gamma e^3 = 0$, $\alpha,\beta,\gamma \neq 0$

$(\gamma+\beta, \alpha, \beta, \alpha) = (0,0,0,0)$

0 0 0 0 0 0

$\gamma + 0 = 0 \Rightarrow \gamma = 0$

$\alpha = \beta = \gamma = 0$ is only solution.

$\Rightarrow$ L.I.

$$A \oplus B = \begin{pmatrix} a_{11} - b_{11} & a_{12} b_{22} \\ a_{21} + b_{12} - 3 & b_{21} - a_{22} + 1 \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$C \oplus D$, $C = A \oplus B$

$$(A \oplus B) \oplus D = C \oplus D$$

$$= A \oplus (B \oplus D)$$

$$A \oplus F = \begin{pmatrix} c_{11} - d_{11} & c_{12} d_{22} \\ c_{21} + d_{12} - 3 & d_{21} - c_{22} + 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} - d_{11} & a_{12} b_{22} d_{22} \\ a_{21} + b_{12} - 3 + d_{12} - 3 & d_{21} - b_{21} + a_{22} + 1 - 1 \end{pmatrix}$$

$$A \oplus B = A^T + B \qquad (A \oplus B)$$

$$(A \oplus B) \oplus C$$

$$\underbrace{(A^T + B)}_{D = A^T + B} \oplus C = D \oplus C = D^T + C$$

$$= (A^T + B)^T + C$$

$$= A + B^T + C$$

PMT 1) $S = \{(x_1, x_2, x_3) \mid x_1 + 3x_2 - 5x_3 = 0\}$

$$x_1 = (3, -1, 0)$$
$$x_2 = (5, 0, 1)$$
$$x_3 = (0, 1, \tfrac{3}{5})$$

all 3 solve

$$\Rightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

$$3\alpha_1 - 3\alpha_1 + 5\alpha_2 - 5\alpha_2 + 3\alpha_3 - 3\alpha_3 = 0$$
$$\forall \; \alpha_1, \alpha_2, \alpha_3$$

$$\langle \{(3, -1, 0), (5, 0, 1), (0, 1, 3/5)\} \rangle$$
$$\subset S$$

$\forall$ arbit

$$\alpha x_1 + \beta x_2 + \gamma x_3 = (a, b, c)$$

$$\begin{aligned} 3\alpha + 5\beta \phantom{+ \tfrac{3}{5}\gamma} &= a \\ -\alpha + \gamma &= b \\ \beta + \tfrac{3}{5}\gamma &= c \end{aligned}$$

$$\sim \left(\begin{array}{ccc|c} 3 & 5 & 0 & a \\ -1 & 0 & 1 & b \\ 0 & 1 & 3/5 & c \end{array} \right)$$

RREF

$S \subset \langle \{ \dots \} \rangle$

$\Rightarrow \quad S = \langle \{ \dots \} \rangle$