Lecture 8: 07/10/20

- We've solved systems of equations of the form $Ax = b$,

  where $A \in M_{m \times n}$, $x \in M_{n \times 1}$

  $b \in M_{m \times 1}$

→ look at v.s. interpretation

$$Ax = b, \quad x = x_p + x_N$$

particular solution

solution from nullspace

$$Ax_p = b, \quad \underline{Ax_N = 0}$$

$$A(x_p + x_N) = Ax_p + Ax_N = b + 0$$
$$= b$$

- Given a matrix $A \in M_{m \times n}$ the rank of A, denoted Rank A, is the number of pivots in RREF.

Ex: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ rank 2 (2 pivots)

$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$ rank 2

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ rank 1 (not a pivot)

$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$ rank 3

- The column space: subset of $\mathbb{R}^m$ which is spanned by the columns of A.

Def: Given $A \in M_{m \times n}(\mathbb{R})$, $A = (a_1 \ a_2 \ \cdots \ a_n)$ where each $a_j \in M_{m\times 1}(\mathbb{R})$, the column space of A, denoted $C(A)$ is the span of the columns; i.e.

$$C(A) = \langle \{ a_1, a_2, \ldots, a_n \} \rangle$$

$C(A)$ contains all vectors Ax for any $x \in M_{n\times 1}(\mathbb{R})$. This means $Ax = b$ is solvable when $b \in C(A)$.

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad Ax = b$ is solvable for any $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\begin{bmatrix} x = b_1 \\ y = b_2 \end{bmatrix} \quad b_1, b_2 \in \mathbb{R}$$

$$\left\langle \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right\rangle \cong \mathbb{R}^2$$

$$\langle \{ (1,0), (0,1) \} \rangle = \mathbb{R}^2$$

Ex: $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $x \in M_{3\times 1}(\mathbb{R})$

what $b \in M_{2\times 1}(\mathbb{R})$ has a solution

$$\langle \{ (1,0), (0,1), (1,1) \} \rangle = \mathbb{R}^2$$

$\Rightarrow \forall\, b \in M_{2\times 1}(\mathbb{R})\ \exists$ a solution

$$M_{3\times 1} \cong M_{1\times 3} \cong \mathbb{R}^3$$

Ex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = A$ $\quad x \in M_{2\times 1}(\mathbb{R})$

$b \in M_{3\times 1}$

$$\underbrace{\langle \{ (1,0,1), (0,1,1) \} \rangle}_{\text{dimension is at most 2}} \neq \mathbb{R}^3$$

L.I.

$\dim \mathbb{R}^3 = 3$

b is a solution to $Ax = b$ if and only if

$$\boxed{b = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

$b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ has a solution $\alpha = 1, \beta = 1$

but $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ doesn't

Denoted $N(A)$

- Nullspace:

Given $A \in M_{m \times n}(\mathbb{R})$, the subspace of $\mathbb{R}^n$ of solutions to

$$\underline{Ax = 0}$$

So $\bar{0}$ always in $N(A)$, the trivial nullspace is <u>only</u> $\bar{0}$, the trival nullspace has dimension 0.

- If $m < n$, then the dimension of the nullspace is non-zero

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right) \quad m = 2 < n = 3$$

RREF

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \alpha_1 & 0 \\ 0 & 1 & \alpha_2 & 0 \end{array}\right) \qquad \begin{aligned} x &= -\alpha_1 z \\ y &= -\alpha_2 z \end{aligned}$$

Free variable

$$\boxed{(-\alpha_1, -\alpha_2, 1)}$$

- Correspondence between non-pivot columns, & free variables

- Terminology: we say a non-pivot column is a free column.

Dimension of nullspace is the # of free columns

Ex. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ so $\dim N(A) = 2$

pivots

2 free columns

$x = -w$
$y = -2z$ } 2

$N(A) \subset \mathbb{R}^n$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right)$$

$x = 0$

$y = -2z$ ← second free

$0w = 0$ → one free

Thm (Rank-Nullity Theorem): Given $A \in M_{m \times n}$

$$\dim N(A) + \text{Rank } A = n$$

number of columns

\# of unknowns

\# of free variables

\# fixed by A

Ex: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = A$

Rank A = 2

$n = 3$

$\Rightarrow \dim N(A) = 1$

one free variable

$\dim \langle \{(0,0,0,\dots,0)\} \rangle = 0$

$\langle \dots \rangle = \{(0,0,\dots,0)\}$ single point

$$S = \{(x_1, x_2, x_3) \mid x_1 + 3x_2 - 5x_3 = 0\}$$

a_1, a_2, a_3

$(0, 1, 3/5), (5, 0, 1)$

$(3, -1, 0)$

0

1

3/5

~ RREF gives rank

x 4 3×4 matrix

$$3\begin{pmatrix} 1 & 0 & \sim & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & 0 \quad 0 & 0 \end{pmatrix}$$

Rank 2, $\Rightarrow N(A) = 2$

2 free parameters

Rank $A = r$

$\downarrow$

Def: Full column rank means $r = n$

$\Rightarrow \dim N(A) = 0$

Full row rank $r = m \Rightarrow Ax = b$ always solvable i.e. $C(A) = \mathbb{R}^m$.

- Four possibilities depending on rank

$A \in M_{m \times n}(\mathbb{R})$

$r = m$ and $r = n$ square and $\forall\ b \in M_{m \times 1}$ there is a unique solution

$r = m$ and $r < n$ $Ax = b$ has ∞ many solutions for each b

$r < m$ and $r = n$ $Ax = b$ has either 1 or 0 solutions for each b

$r < m$ and $r < n$ $Ax = b$ has either 0 or ∞ solutions

$r < m$ & $r < n$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{spanned by columns}} \quad \text{rank } 2$$

b ask has a unique sol

$\Rightarrow$ if $Ax = b$ has a solution it is not unique

$$\langle \{ (1,0,0), (0,1,0) \} \rangle.$$

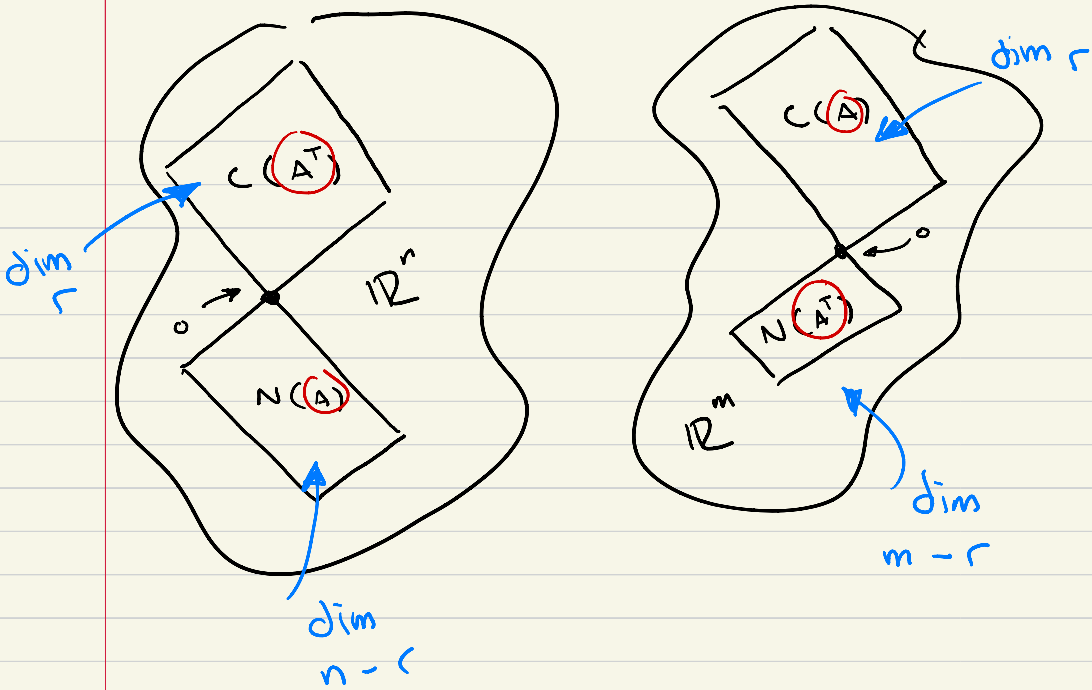
- Row space $C(A^T) \subset \mathbb{R}^n$
- Left nullspace $N(A^T) \subset \mathbb{R}^m$

$$\dim C(A^T) = r$$

$$\dim N(A) = n - r$$

$$\dim C(A) = r$$

$$\dim N(A^T) = m - r$$



- Fundamental theorem of linear algebra

$\dim C(A^T) = \dim C(A) = r = \text{Rank } A$

$\dim N(A) = n - r$

$\dim N(A^T) = m - r$

$S = \{(x_1, x_2, x_3) \mid \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b\}$

1 $\quad S = \{(x_1, x_2, x_3) \mid x_1 + 3x_2 - 5x_3 = 0\}$

Note that $\{(1,0,0), (0,1,0), (0,0,1)\}$ doesn't work $\to$ $\mathbb{R}^3 \subset S$

$1 = 0 \qquad 3 = 0 \qquad -5 = 0$ $\quad \updownarrow$

not true $\qquad S \subset \mathbb{R}^3$

$x_1 + 3x_2 - 5x_3 = 0$

0	1	3/5			
5	0	1	10	0	2
-3	1	0			

$M = \{(0, 1, 3/5), (5, 0, 1), (-3, 1, 0)\}$

$\langle M \rangle \subset S$ because linear combos are still solutions

$\alpha(0, 1, 3/5) + \beta(5, 0, 1) + \gamma(-3, 1, 0)$

$\underline{\underline{= 0}} \quad \Rightarrow \quad \underline{\langle M \rangle \subset S}$

$\langle M \rangle = S$, $\langle M \rangle \subset S$ and $S \subset \langle M \rangle$

$J = \{(-3, 1, 0)\}$

$\langle J \rangle \subset S$, $\langle J \rangle \not\supset S$

$a + 3b - 5c = 0$

Given $(a, b, c) \in S$, want to show

$(a, b, c) \in \langle M \rangle$

$\Rightarrow \quad S \subset \langle M \rangle$

unknown is (α, β, γ)

$$\begin{aligned} 0\alpha + 5\beta - 3\gamma &= a \\ \alpha + 0\beta + \gamma &= b \\ \tfrac{3}{5}\alpha + \beta + 0\gamma &= c \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 5 & -3 & b \\ 3/5 & 1 & 0 & c \end{array}\right)$$

$R_2 = \frac{1}{5} R_2$

$R_3 = R_3 - \frac{3}{5} R_1$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & -3/5 & a/5 \\ 0 & 1 & -\frac{3}{5} & c - \frac{3}{5} b \end{array}\right)$$

2 pivots

$R_3 = R_3 - R_2$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -3/5 & b/5 \\ 0 & 0 & 0 & c - \frac{3}{5} b - \frac{a}{5} \end{array}\right)$$

$$c - \frac{3}{5} a - \frac{b}{5} = 0$$

$n = 3$

$\Rightarrow$ 1 free variable

$$\boxed{a + 3b - 5c = 0}$$

this has one free variable

$$\begin{cases} \alpha + \gamma = a \\ \beta - \frac{3}{5}\gamma = \frac{b}{5} \end{cases}$$

$\Rightarrow$ we get a solution

$\gamma = 1$, fixes choice of α, β

So $\alpha = a - \gamma$

$\beta = \frac{b}{5} + \frac{3}{5}\gamma$

infinitely many ways to form (a, b, c)

$\Rightarrow$ $S \subset \langle M \rangle$

$S = \langle M \rangle$, M a spanning set of S

$A \subset B \quad B \subset A \Rightarrow A = B$

$a \leq b \quad b \leq a \Rightarrow a = b$

Def: M a spanning set of S if and only if

$\langle M \rangle = S$

$\langle M \rangle = S$ if and only if

$\langle M \rangle \subset S$ & $S \subset \langle M \rangle$

We proved

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Sd

in $\mathbb{R}$ there is only one zero

and $\forall\ \alpha \in \mathbb{R}$ $\quad \alpha + 0 = \alpha$

$$\alpha + 0_{\alpha}^{=0} = \alpha$$

$$\alpha + (-\alpha) = 0$$

no zero

$$A \oplus (-A) = 0$$

$D \in Z$, $\forall\ M \in M_{m \times m}(\mathbb{R})$

$$DM = MD$$

$$D = (A + B)$$

$$(A + B)C$$

$$= AC + BC$$

bc $A, B \in Z$

$$= CA + CB$$

$$= C(A+B)$$

$$\Rightarrow (A+B) \in \mathbb{Z}$$

2a $\sin x \neq \alpha \cos x$ for any α

if $\alpha \neq 0$, choose $x = 0$

$$\sin 0 = 0 \neq \alpha = \cos 0$$

$\alpha = 0$, choose $x = \frac{\pi}{2}$

$$1 = \sin \frac{\pi}{2} \neq 0 = \alpha \cos\left(\frac{\pi}{2}\right)$$

scalar mult

$(V, \mathbb{F}, \oplus, \otimes)$

this operation is matrix mult.