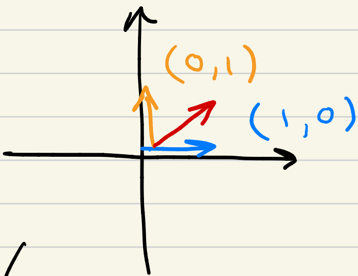
Lecture 9: 07/13/20

- The Determinant

A new way to look at a matrix (square matrix).



"unit vectors"

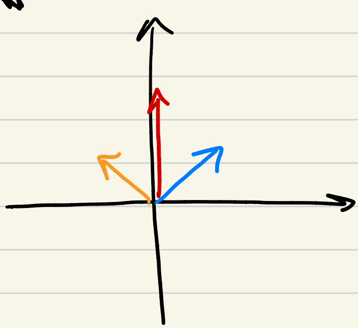
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Linear transformation

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

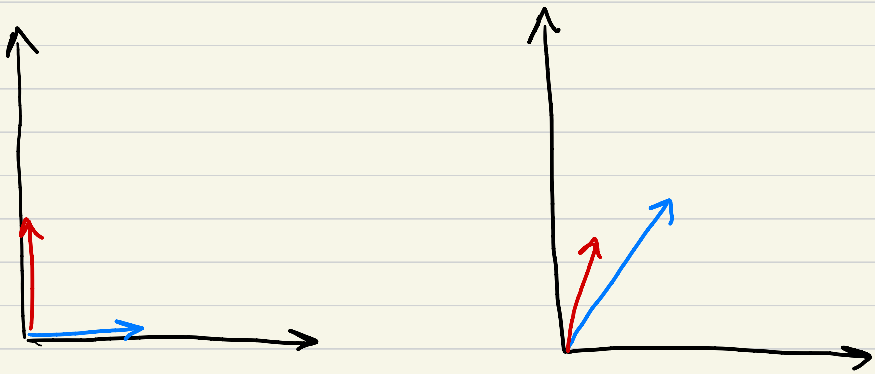
$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

$v \in M_{2\times1}(\mathbb{R})$



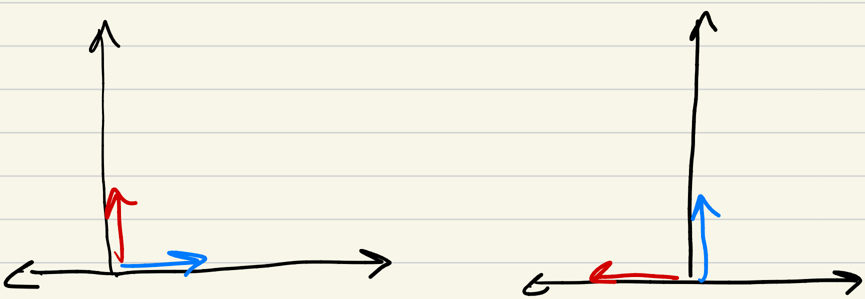
Ex:

$$A = \begin{pmatrix} 1 & 1/2 \\ 3 & 2 \end{pmatrix}$$



Ex:

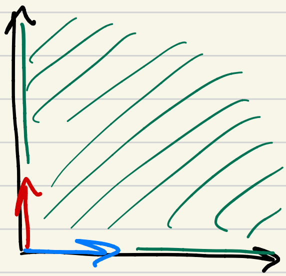
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

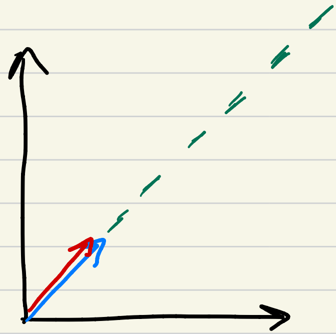


$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

Ex $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

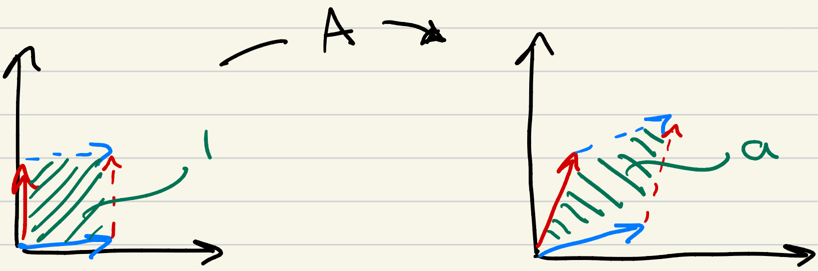




L9 Q1: pw: mat

- A good heuristic for the determinant is it measures area in $\mathbb{R}^2$ & volume in $\mathbb{R}^3$, n-dimensional volume in $\mathbb{R}^n$.

- 5-B: go from scratch



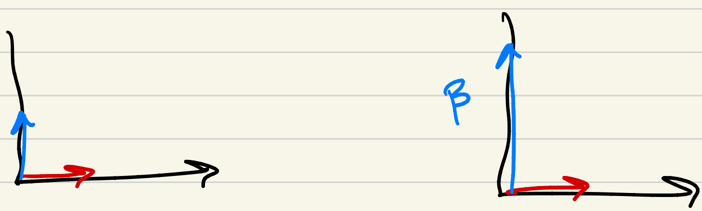
$$\det A = a$$

- 3 Properties of a determinant

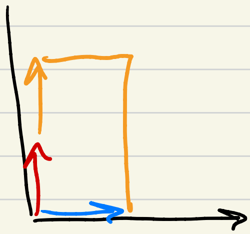
(1) Linear in rows

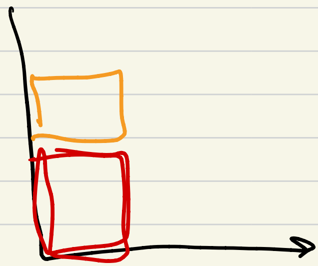
b_i be row vectors

$$\det \begin{pmatrix} b_1 \\ \vdots \\ \beta b_j \\ \vdots \\ b_n \end{pmatrix} = \beta \det \begin{pmatrix} b_1 \\ \vdots \\ b_j \\ \vdots \\ b_n \end{pmatrix}$$



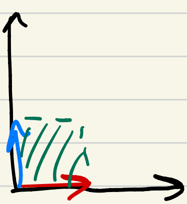
$$\det\begin{pmatrix} b_1 \\ \vdots \\ b_j + a \\ \vdots \\ b_n \end{pmatrix} = \det\begin{pmatrix} b_1 \\ \vdots \\ b_j \\ \vdots \\ b_n \end{pmatrix} + \det\begin{pmatrix} b_1 \\ \vdots \\ a \\ \vdots \\ b_n \end{pmatrix}$$



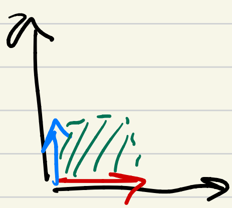


② A matrix w/ two identical rows to have $\det A = 0$.

③ $\det I = 1$



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



- 3 big questions:
  - → Does a function satisfying ①-③ exist?
  - → If it exists, is it unique?
  - → If so, how do we (practically) compute?
- Permutations:

  Consider the first n integers

  $$1, 2, 3, \ldots, n$$

  a permutation is an <u>ordered</u> arrangement of $1, 2, \ldots, n$.

Ex: $n = 3$

$$(3, 2, 1), (1, 2, 3), (2, 1, 3)$$

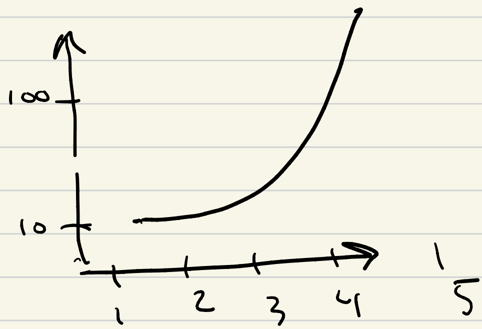
are all permutations

→ There are $n!$ permutations of $1, 2, \ldots, n$.

$$n! = n(n-1)(n-2)\cdots 2\cdot 1$$

$$\underset{1}{\underline{n}} \times \underset{2}{\underline{n-1}} \times \underset{3}{\underline{n-2}} \times \cdots \times \underset{n-1}{\underline{2}} \times \underset{n}{\underline{1}} = n!$$

- Note: $n!$ gets really big really fast



$(1, 2, 3) \neq (2, 1, 3)$

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720

7 5040

Originally studed by Evariste Galios

- Finally, we call the set of all permutations the symmetric group or the permutation group, denoted by S_n.

$S_1 = \{(1)\}$

$S_2 = \{(1,2), (2,1)\}$

$S_3 = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

- Q2, PW: Sym

- We say two elements in a permutation are inverted if they are out of their natural order:

$p_1 = 3$

$(p_1, p_2, \ldots, p_n)$

$p_i > p_k$, $i < k$, then p_i & p_k are inverted

$(1, 3, 2)$
p_1 p_2 p_3

$p_2 > p_3$, $2 < 3$

$\Rightarrow$ p_2 & p_3 are inverted

- An inversion will swap inverted elements

$(1, 3, 2) \xrightarrow{\text{inversion}} (1, 2, 3)$

Inversion only swaps adjacent

- $N(p_1, p_2, \ldots, p_n)$ = # of inverted elements

= # of inversions to get to

$(1, 2, 3, 4, \ldots, n)$

3 inverted elements

$(3,2,1,4) \overset{1}{\rightarrow}$ $N(3,2,1,4) = 3$

$(2,3,1,4) \overset{2}{\rightarrow}$ $N(2,3,1,4) = 2$

$(2,1,3,4) \overset{3}{\rightarrow}$ $N(2,1,3,4) = 1$

$(1,2,3,4)$ $N(1,2,3,4) = 0$

- Classify a permutation as even or odd depending on if N is even or odd

  if N odd, the permutation has odd pariety

  " even, " " even

- $\sigma(p_1, p_2, \dots, p_n)$ (sign function)

$$\sigma(p_1, p_2, \dots, p_n) = (-1)^{N(p_1, p_2, \dots, p_n)}$$

$$\sigma = \begin{cases} 1 & \text{if } (p_1, p_2, \dots, p_n) \text{ is even} \\ -1 & \text{if } \dots \text{ odd} \end{cases}$$

Def: Given $A \in M_{n\times n}(\mathbb{R})$, $A = (a_{ij})_{n\times n}$,

$$\det A = \sum_{(p_1,p_2,\dots,p_n)\in S_n} \sigma(p_1,p_2,\dots p_n)\, a_{1p_1} a_{2p_2} \cdots a_{np_n}$$

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $a_{11} = 1$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = 1$

$2\times 2 \Rightarrow n = 2 \Rightarrow S_n = \{(1,2),(2,1)\}$

$\sigma(1,2) = 1$, $\sigma(2,1) = -1$

$N(1,2) = 0 \quad N(2,1) = 1$

$$\det A = \sigma(1,2)\, a_{11} a_{22} + \sigma(2,1)\, a_{12} a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

$$= (1)(1) - (0)(0) = 1$$

Ex: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $a_{11} = 1$, $a_{12} = 1$, $a_{21} = 1$, $a_{22} = 1$

$$\det A = \sigma(1,2)\, a_{11} a_{22} + \sigma(2,1)\, a_{12} a_{21}$$

$$= (1)(1) - (1)(1) = 0$$

Ex: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det A = \sigma(1,2)\, a_{11} a_{22} + \sigma(2,1)\, a_{12} a_{21}$$

$$= ad - bc$$

a_{kp_k} = kth row, p_kth column

S_n

$\frac{n}{2}$ even, $\frac{n}{2}$ odd

Ex: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$S_3 = \{ \overset{N,\ \sigma}{\ } \overset{0,\ 1}{(1,2,3)},\ \overset{1,\ -1}{(1,3,2)},\ \overset{1,\ -1}{(2,1,3)},\ \overset{2,\ 1}{(2,3,1)},\ \overset{2,\ 1}{(3,1,2)},\ \overset{3,\ -1}{(3,2,1)} \}$$

$$\prod_{k=1}^{n} a_{kp_k}$$

$$a_{2p_2}a_{1p_1}a_{3p_3}$$

$$\shortparallel$$

$\pi = (p_1, p_2, \ldots, p_n)$ $\quad a_{1p_1}a_{2p_2}a_{3p_3}$

$$\sum_{\pi \in S_3} \sigma(\pi)\, a_{1p_1} a_{2p_2} a_{3p_3} = \det A$$

$$\overset{1}{\sigma(1,2,3)}\, a_{11}a_{22}a_{33} + \overset{-1}{\sigma(1,3,2)}\, a_{11}a_{23}a_{32}$$
$$+ \overset{-1}{\sigma(2,1,3)}\, a_{12}a_{21}a_{33} + \overset{1}{\sigma(2,3,1)}\, a_{12}a_{23}a_{31}$$
$$+ \overset{1}{\sigma(3,1,2)}\, a_{13}a_{21}a_{32} + \overset{-1}{\sigma(3,2,1)}\, a_{13}a_{22}a_{31}$$

$$\boxed{\begin{aligned} \det A &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}}$$

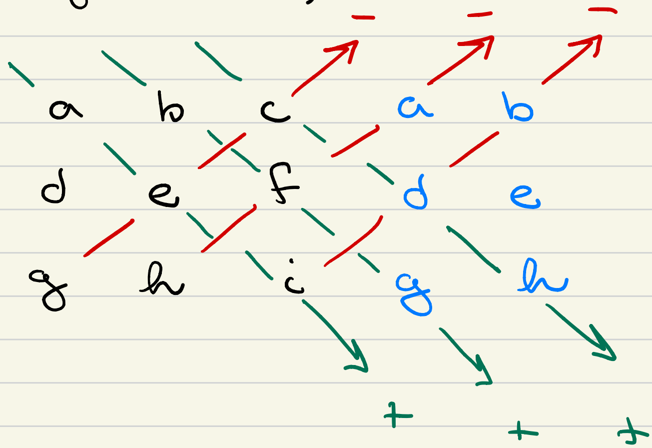
$p \in S_n$, $p = (p_1, p_2, \ldots, p_n)$

$$\sigma(p)\, a_{1p_1} \cdots a_{np_n}$$

Spelling?

- A practical algorithm

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$



$$aei + bfg + cdh - gec - hfa - idb$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} = A$$

$$\begin{array}{ccccc} & & - & - & - \\ 1 & 0 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 1 & 2 & -1 & 1 & 2 \\ & & + & + & + \end{array}$$

$$= 1 + 0 + 0 + 3 - 4 + 0$$

$$= 1 - 1 = 0$$

$\det A = 0$

PW: easy

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

(1) write first two columns to the right

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

(2) arrows 3 long: memorize $\mp$

$$\begin{matrix} & & - & - & - \\ a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ & & + & + & + \end{matrix}$$

③ Take product along arrow, add all products

$$aei + bfg + cdh - gec - hfa - idb.$$

END OF LEC ↑

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} a$$

$ei - hf$ $\qquad$ $di - fg$ $\qquad$ $dh - eg$

$$a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= aei - ahf - bdi + bfg + cdh - ceg$$

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

$W \subset V$

HW3-2 $\quad \dim W = n \overset{= \dim V}{} \iff V = W$

$V = W \Rightarrow \dim W = \dim V$

$\dim W = \dim V \Rightarrow V = W$

V is a v.s. so it has a base

$\{v_1, v_2, \dots, v_n\}$ n elements

$\Rightarrow \dim V = n$

$W \subset V$, W a subspace $\Rightarrow W$ is a v.s.

$\Rightarrow W \quad \{w_1, w_2, \dots, w_m\}$ don't need

$V \subset W \Rightarrow \underbrace{v_1, v_2, \dots, v_n}_{\text{L.I.}} \in W$

W has n or more dimensions

$\dim W \geq n$

$W \subset V \Rightarrow \underline{\underline{\dim W \leq \dim V}}$

$\dim W = \dim V$

$\dim W = \dim V \Rightarrow W = V$

choose base of W $\{w_1, w_2, \ldots, w_n\}$ S

and v $\{v_1, v_2, \ldots, v_n\}$

$W \subset V$ $\underbrace{\{w_1, w_2, \ldots, w_n\}}_{n \text{ element l.I. set}} \subset V$

so $\{w_1, w_2, \ldots, w_n\}$ is a base for V

We chose S $\cdot\ni\cdot$ $\langle S \rangle = W$

we showed $\langle S \rangle = V$

$\Rightarrow \langle S \rangle = \underline{\underline{V = W}}$ ▨

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ 0 & \frac{a_{22}}{a_{22}} & \frac{a_{23}}{a_{33}} \\ 0 & 0 & \end{pmatrix} \overset{RREF}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{L.I.}$$

1) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$ $a_{ii} \neq 0$

RREF to I

(a) Show columns are L.I.

$$\overset{a}{(a_{11}, 0, 0)}, \overset{b}{(a_{12}, a_{22}, 0)}, \overset{c}{(a_{13}, a_{23}, a_{33})}$$

$\alpha a + \beta b + \gamma c = 0$ only if $\alpha, \beta + \gamma = 0$

$$(\alpha a_{11} + \beta a_{12} + \gamma a_{13}, \beta a_{22} + \gamma a_{23}, \gamma \underbrace{a_{33}}_{\text{non-zero}}) = (0, 0, 0)$$

$\Downarrow$

$\gamma = 0$

$$(\alpha a_{11} + \beta a_{12}, \beta \underbrace{a_{22}}_{\text{non-zero}}, 0)$$

$\Downarrow$

$\beta = 0$

$(\underbrace{\alpha a_{11}}_{\text{non-zero}}, 0, 0)$

$\alpha = 0$

$\Rightarrow \alpha = \beta = \gamma = 0$

and cols are L.I.

$$\left(\begin{array}{ccc|c} a_{11} & & & \alpha \\ & \ddots & & \beta \\ & & \ddots & \gamma \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

$a_{11}, a_{22}, a_{33} \neq 0$

$$a(a_{11}, 0, 0) + b(a_{12}, a_{22}, 0) + c(a_{13}, a_{23}, a_{33})$$

$$= \underline{\underline{(\alpha, \beta, \gamma)}}$$

tells us we span.

$$\sum_{k=1}^{n} \alpha_k \sum_{j=1}^{k} (a_{1k}, \dots)$$

$$\alpha_{n-1} a_{n-1\,n} \qquad , \alpha_n a_{nn}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdots & a_{1m} \\ 0 & a_{22} & a_{23} & & & a_{2m} \\ 0 & 0 & a_{33} & & & a_{3m} \\ 0 & & & \ddots & & \vdots \\ 0 & & & & \ddots & \vdots \\ 0 & & \cdots & \cdot & 0 & a_{mm} \end{pmatrix}$$

k-th column

Let $\sum_{k=1}^{m} \alpha_k a_k = 0$

then $\alpha_m a_{mm} = 0$

$\Rightarrow \alpha_m = 0$

then $\alpha_{m-1} a_{m-1\,m-1} + \underbrace{\alpha_m}_{=0} a_{m-1\,m} = 0$

$\alpha_{m-1} \underbrace{a_{m-1\,m-1}}_{\text{non-zero}} = 0$

$\Rightarrow \alpha_{m-1} = 0$

continuing shows

$\alpha_k = 0 \ \forall \ k$

$\Rightarrow$ L.I.

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$ cols L.I.

we assume true for an $m \times m$ and show that it is also true for $m+1 \times m+1$