Lecture 10 : 07/15

PW: det

- Minors: Given a matrix $A \in M_{m \times m}(\mathbb{R})$

The minor Γ_{rs} is the determinat of the matrix obtained by removing rth row and sth column.

Ex
$$\begin{pmatrix} 1 & 0 & 4 & 5 \\ -1 & 1 & 3 & -2 \\ 6 & 2 & -1 & 4 \\ 3 & -2 & 1 & 7 \end{pmatrix}$$

Γ_{23}

Γ, γ

$$\Gamma_{23} = \det \begin{pmatrix} 1 & 0 & 5 \\ 6 & 2 & 4 \\ 3 & -2 & 7 \end{pmatrix}$$

Ex $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\Gamma_{11} = \det(d) = d$

- Cofactors:

$$C_{ij} = (-1)^{i+j} \Gamma_{ij}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \boxed{a}\boxed{d} - bc$$

$$C_{11} = d$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\boxed{a_{22}} - a_{12}\boxed{a_{21}}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$a_{22}a_{33} - a_{32}a_{23}$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32})$$

$$a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23}$$

cofactor of a_{11} is

$$\boxed{a_{22}a_{33} - a_{23}a_{32}}$$

- Cofactor Expansion

$$\begin{pmatrix} 1 & 0 & 5 \\ 6 & 2 & 4 \\ 3 & -2 & 7 \end{pmatrix}$$

Cofactor expansion Theorem $A \in M_{m \times m}$

$$\det A = \sum_{k=1}^{m} a_{ik} C_{ik}$$

Proved in 6-B

for any $1 \le i \le m$

a_{11}, a_{12}, a_{13}

C_{11} C_{12} C_{13}

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$C_{11} = 22$, $C_{12} = -30$, $C_{13} = -18$

$$22 + 0(30) + 5(-18) = 22 - 90$$

$$= -68$$

pw: minor

$$\begin{pmatrix} 1 & 0 & 5 \\ 6 & 2 & 4 \\ 3 & -2 & 7 \end{pmatrix}$$

$i = 2$ $\quad a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$C_{21} = -10, \quad C_{22} = -8, \quad C_{23} = 2$

$6(-10) + 2(-8) + 4(2)$

$-60 - 16 + 8 = -68$

- $\det(A) = \det(A^T)$

$\det A = |A|$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$+ \underset{0}{a_{12}} c_{12} + \underset{0}{a_{13}} c_{13}$

$\underset{1}{a_{11}} C_{11} + \underset{-1}{a_{14}} C_{14} = C_{11} - C_{14}$

$$C_{11} = \begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 2C^*_{11} + 3C^*_{12} = 5$$

$C^*_{11} = 1$

$C^*_{12} = -(-1) = 1$

$$\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$$

$$C_{14} = - \begin{vmatrix} \boxed{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} \boxed{2 \quad 3} \\ 0 \quad 2 \\ \boxed{1 \quad 3} \end{matrix} \end{vmatrix} = (-1)(-1)\left(3\right)$$

$= 3$

$\det A = 5 - 3 = 2$

- key takeaways

look for easy rows & columns

$$\sum_{\boxed{\pi \in S_n}} \sigma(\pi)\, a_{1\pi_1} \cdots a_{n\pi_n} \qquad |S_n| = n!$$

$1 \,\_\_ \quad 2 \,\_\_ \quad 3 \,\_\_ \quad 4 \,\_\_ \qquad 4! = 24$

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 1 & 2 & 3 \\
4 & 5 & 6 & 7 & 4 & 5 & 6 \\
-7 & 8 & -9 & 1 & -7 & 8 & -9 \\
2 & 3 & 4 & 5 & 2 & 3 & 4
\end{array}$$

$5 \quad 6 \quad 7 \quad 8$

Thm: $A \in M_{m \times m}(\mathbb{R})$, A is invertible (= nonsingular) if and only if $\det A \neq 0$.

$\Rightarrow$ If A is invertible $A \sim I$

$\det A$ = non-zero multiple of $\det I$

G-B, Strang

$\det A^*$, where $A \sim A^*$.

$\det A = \alpha^* \det A^*$

$\alpha^* \neq 0$, $A \sim A^*$

$\Rightarrow \det A \neq 0$

$\Leftarrow$ $\det A \neq 0$, then $A \sim I$ (RREF)

$\Rightarrow$ A is invertible

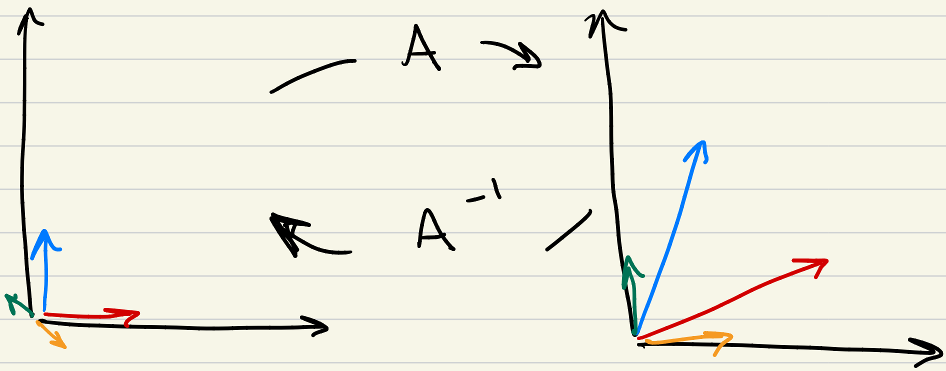
$$\underbrace{\begin{pmatrix} a_{11} & \cdots & \vline & a \\ \vdots & & \vline & b \\ \vdots & & \vline & c \\ & & \vline & \vdots \end{pmatrix}}_{\sim I}$$

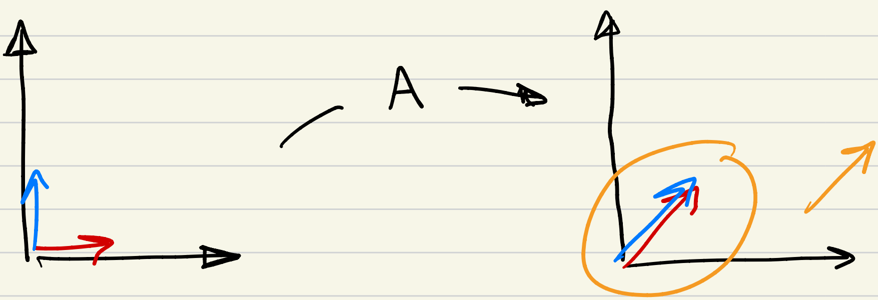
unique solution

$$\begin{pmatrix} a_{11} & \cdots & \vline & 1 & 0 & 0 \\ \vdots & & \vline & 0 & 1 & \\ \vdots & & \vline & & & \ddots \end{pmatrix}$$

$$\sim I$$

$\blacksquare$





pw : inv

$A^{-1} \nearrow$

$= \rightarrow \quad ?$

$\neq$

$= \uparrow$

Linear Transformation

- Mapping between v.s.

Let V, W be v.s. a mapping T (or transformation T) is a function

$T: V \to W$, $\forall\ v \in V\ \exists$ a unique $w \in W \ni$.

domain range

$$T(v) = w$$

$T(v) = Tv$ ← we drop parens.

it is not always true, that

not always true

$\forall\ w \in W\ \exists\ v \in V \ni$.

$$Tv = w$$

$\forall\ w \in W\ \exists$ only one $v \ni$.

$$Tv = w$$

- **Linearity**: $T : V \to W$ is linear if

  1. $T(u+w) = Tu + Tw, \quad \forall\, u, w \in V$
  2. $T(\alpha u) = \alpha Tu, \quad \forall\, \alpha \in \mathbb{R}.$

Consequence:

Recall that in a v.s. $\underset{\in \mathbb{F}}{0}\,v = \underset{\in V}{\bar{0}}$

$$T(\bar{0}) = T(0v) = 0\,\underbrace{Tv}_{w \in W} = \underset{\in \mathbb{F}}{0}\,w = 0$$

$$\left.\begin{aligned} T(-v) &= -Tv \\ w &= -w \end{aligned}\right\} \text{ "negatives" map to "negatives"}$$

(x_1, x_2)

Ex: $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T(x) = (2x_1 + x_2,\ 3x_1 - x_2,\ -5x_1 + 3x_2)$$

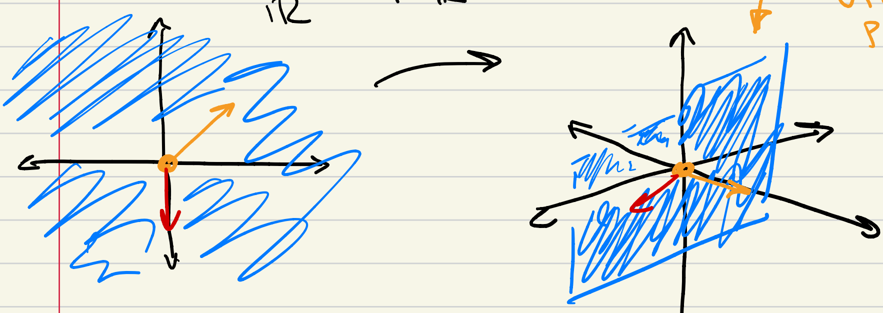
$x + y$

$$T(x+y) = T(x) + T(y)$$

$$T(x+y) = \begin{pmatrix} 2(x_1+y_1) + x_2 + y_2, \\ 3(x_1+y_1) - (x_2+y_2) \\ -5(x_1+y_1) + 3(x_2+y_2) \end{pmatrix}$$

$$= T(x) + T(y)$$

$\mathbb{R}^2$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^3$



we can view plane as $\mathbb{R}^2$

PW : lin $\qquad Tv \in W$

$$TV \subset W$$

$$TV = \{ Tv \mid v \in V \}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$a + b + c = 1$$

$$d + e + f = 1$$

$$g + h + i = 1$$

$$a + d + g = 1$$

$$b + e + h = 1$$

$$c + f + i = 1$$

$$a + e + i = 1$$

$$g + e + c = 1$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ -2 * R_1 \\ \rightarrow R_1 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$f: \mathbb{R} \to \mathbb{R}$

$x + 3$

non-linear

"affine"

$$f(x) = x^2 + 3x$$

$$f(5x) = 5f(x)$$

$$f(5x) = 25x^2 + 15x$$

$$\neq 5x^2 + 15x$$

$$= 5f(x)$$

$$v_1, v_2, v_3$$

9 0 4

$$\underline{9 + 4x^2} = \alpha v_1 + \beta v_2 + \gamma v_3$$

want to find α, β, γ

$$(9, 0, 4) = \alpha(1, -1, 1) + \beta(\ldots)$$

$$\begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 4 \end{pmatrix}$$

what are these?

$$\left(\begin{array}{ccc|c} \cdot & \cdot & - & 9 \\ \cdot & - & \cdot & 0 \\ & & & 4 \end{array}\right) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$Ax = b$$

$$\left(\begin{array}{c|c} A & b \end{array}\right) = \underline{\underline{x}}$$

$$\sim$$

$$\underline{x = A^{-1} b}$$