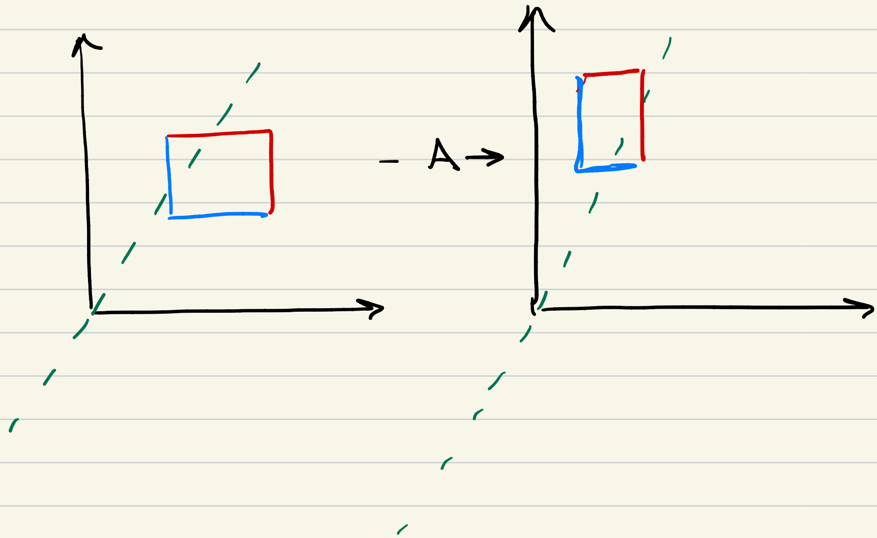
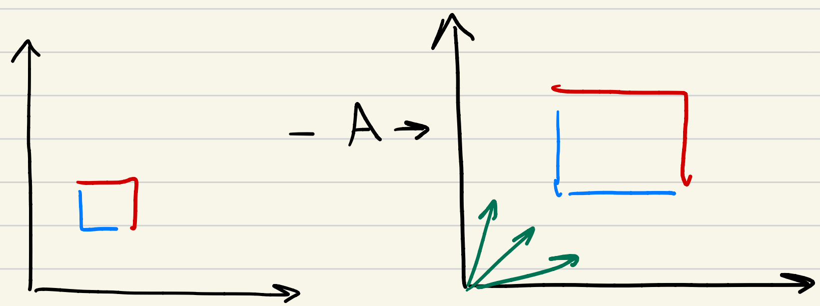
Lecture 11 : 07/17

- The kinds of things linear trans. do (focus on $\mathbb{R}^2$)

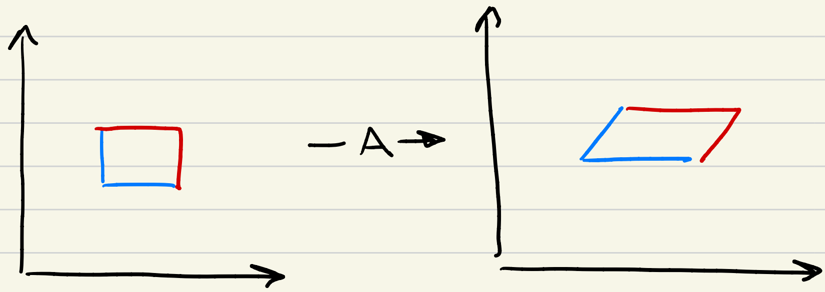
→ Reflections:



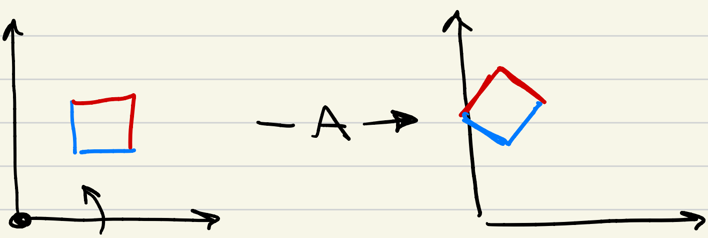
→ Stretch



→ Shearing



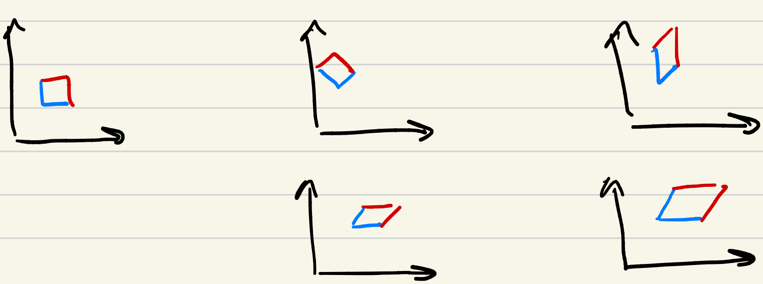
→ Rotation

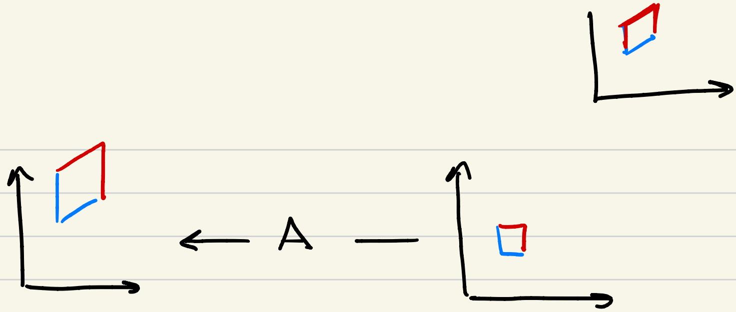


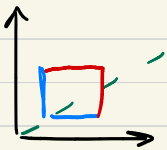
$\quad\quad$ 1 $\quad\quad\quad\quad\quad$ 2 $\quad\quad\quad\quad\quad$ 3

Ex $\quad$ Rotate → shear → stretch

$\quad\quad$ 3 $\quad\quad\quad\quad\quad$ 1 $\quad\quad\quad\quad\quad$ 2



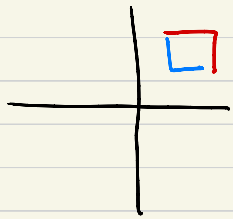




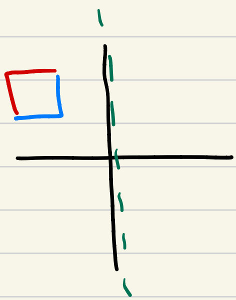
① Reflect → shear

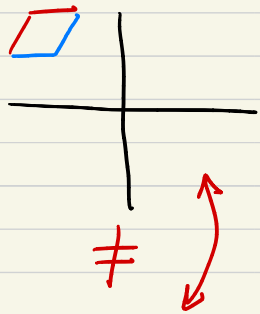
vs

② shear → reflect



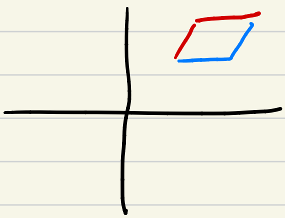
①

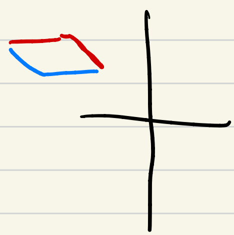




≠

②





- The kernel & the range.

Let $T : V \to W$, kernel of T

$$\ker T = \{ v \in V \mid Tv = 0 \} \subset V$$

Ex $Tx = x^2 - 3$ $\quad T : \mathbb{R} \to \mathbb{R}$

$$Tx = 0 = x^2 - 3 \Rightarrow x = \pm\sqrt{3}$$

$$\ker T = \{ \sqrt{3}, -\sqrt{3} \}$$

Ex: $\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} = A$ find kernel

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sim$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \qquad a + 2b = 0$$

$$\ker A = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \; a + 2b = 0 \right\}$$

Thm: ker T of a linear trans. is a subspace of V.

Pf: Recall subspace theorem. which says it suffices to show ker T is closed under + & ×.

+ let $u, w \in \ker T \Rightarrow Tu = 0,\ Tw = 0$

$T(u+w) = Tu + Tw = 0 + 0 = 0$ ✓

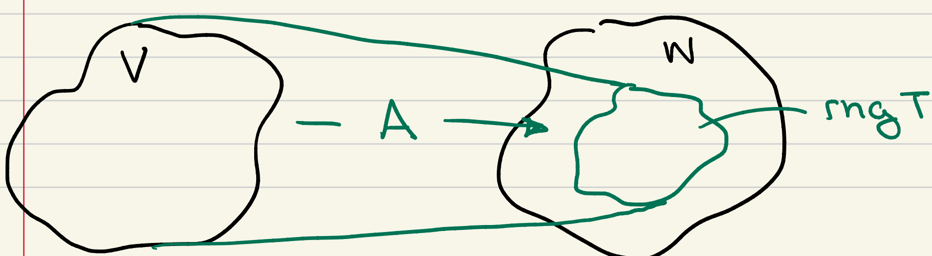
× $\alpha \in \mathbb{R}$ $\quad T(\alpha u) = \alpha Tu = \alpha 0 = 0$ ✓

∎

domain codomain

Def: Let $T: V \to W$ be a L.T. $\operatorname{rng} T \subset W$

$$\operatorname{rng} T = \{ w \in W \mid \exists\, v \in V \text{ s.t. } w = Tv \}$$



$\operatorname{rng} T = TV$

$T(\{\cdots\}) = \{\cdots\}$

$\operatorname{rng} T = \operatorname{Img} T$

Thm: $\operatorname{rng} T$ is a subspace of W

pf: Same theorem

$u, w \in \operatorname{rng} T$, $\exists\ u_v, w_v$ s.t.

$Tu_v = u$, $Tw_v = w$

We want to show $u + w \in \operatorname{rng}$

$\alpha u \in \operatorname{rng}$

$$u + w = Tu_v + Tw_v$$

$$= T(u_v + w_v) \quad \text{by lin.}$$

$u_v + w_v \in V$ since V is a v.s.

$\Rightarrow u + w \in \operatorname{rng}$

$$\alpha u = \alpha T u_v = T(\alpha u_v)$$

V a v.s.

$\blacksquare$

- Relation to matrices

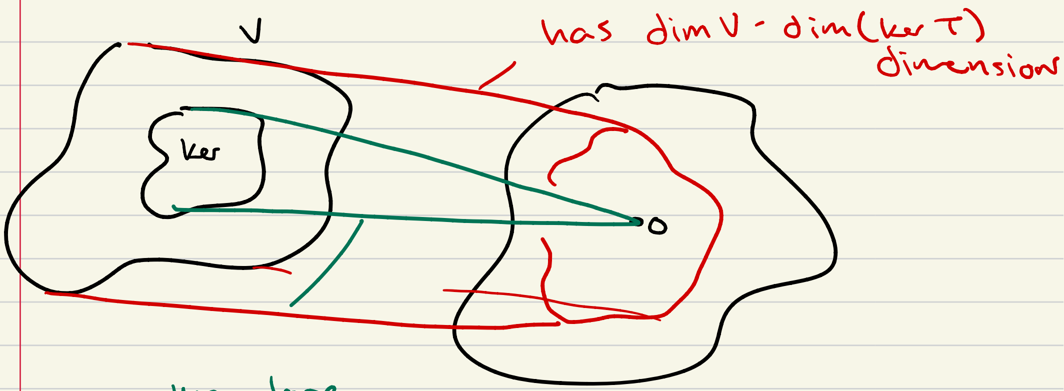
  $\ker A \iff N(A)$ nullspace

  $\operatorname{rng} A \iff C(A)$ column space

Not proved

Thm: (General Rank-Nullity)
$T: V \to W$ a L.T.

$$\dim(\ker T) + \dim(\operatorname{rng} T) = \dim(V)$$



We lose

$\dim(\ker T)$ dimensions

- Composition of linear transformations

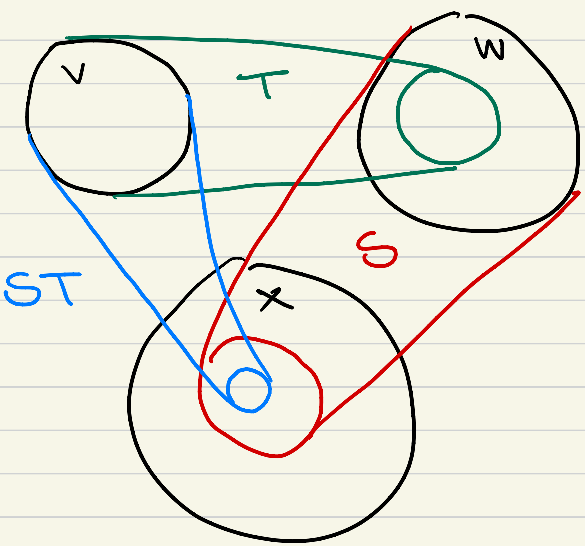
  Let V, W, X be vs.

  $T: V \to W$, $S: W \to X$

  then we define for $v \in V$

  $$\boxed{(ST)}v = Rv := S(Tv)$$

  $$ST : V \to X$$



$$f \circ g : \mathbb{R} \to \mathbb{R}$$

$$\underbrace{f : \mathbb{R} \to \mathbb{R} \qquad g : \mathbb{R} \to \mathbb{R}}$$

Ex $f(x) = x^2$, $g(x) = x + 2$

$$f \circ g(x) = f(g(x)) = x^2 + 4x + 4$$

3×2 $\qquad$ 6×3

Ex $A : \mathbb{R}^2 \to \mathbb{R}^3$, $B : \mathbb{R}^3 \to \mathbb{R}^6$

$$BA : \mathbb{R}^2 \to \mathbb{R}^6$$

$T : V \to W$

- We say a transformation is 1-1, if $\forall$ $w \in W$ there is a unique $v \in V$ $\ni$ $Tv = w$

- We say T is onto if $\operatorname{rng} T = W$.

T a L.T., $T : V \to W$

Thm: $\ker T = \{0\}$ if and only if T is 1-1.

Pf: $\ker T = \{0\} \Rightarrow$ 1-1.

Suppose not, $\exists$ u, w, $\boxed{u \neq w}$ $\ni$ $Tu = Tw$

$Tu = Tw$

$Tu + (-Tw) = Tw + (-Tw)$ exist. neg.

$Tu + (-Tw) = 0$

let $T\varphi = (-Tw)$, $\varphi \in V$

$\to T(-w)$ range is a subspace.

$Tu + T\varphi = 0$ linearity ker $T = \{0\}$

$T(u+\varphi) = 0 \Rightarrow u + \varphi = 0$

$\varphi = -w$

$u - w = 0$

$u - w + w = w$

$u + 0 = w$

$\boxed{u = w}$

$1-1 \Rightarrow \ker T = \{0\}$

Let $\varphi \in \ker T \Rightarrow T\varphi = 0$

Since T is 1-1, then only one

such φ. Since T is linear,

$\varphi = 0$. ∎

No proof

Thm: Let $T: V \to W$ be a linear transformat.
V, W both finite dimensional, then

1. If T is 1-1: $\dim V \leq \dim W$
2. If T is onto: $\dim V \geq \dim W$
3. If T is 1-1 & onto, $\dim V = \dim W$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A, \quad \mathbb{R}^2 \to \{0\} \subset \mathbb{R}^2$$

$A: \mathbb{R}^2 \to \mathbb{R}^2$

is onto for $\{0\}$.

- Inverses. If $T: V \to W$ is 1-1 and onto we define $T^{-1}: W \to V$ by
  $T^{-1}w = v$ where $Tv = w$.

- Linear transformations as matrices.

$T: V \to W$, $\dim V = m$, $\dim W = n$

$\{v_1, v_2, \dots, v_m\}$ Needs to be a base

$\in W$ column vectors of length n

$Tv_1, Tv_2, \dots, Tv_m$

$$A = \begin{pmatrix} Tv_1 & Tv_2 & \cdots & Tv_m \end{pmatrix}$$

$A \in M_{n \times m}(\mathbb{R})$

$$\begin{array}{ccccc} A: & \mathbb{R}^n & \to & \mathbb{R}^m \\ & \cong & & \cong \\ T: & V & \to & W \end{array}$$

A representation of T by A

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$v_1 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad Av_1$$

$$v_2 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad Av_2$$

$$Av_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad Av_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

END OF LEC

WW 1

Solve for X if $AX(D+BX)^{-1} = C$

$$AB(X+C)(D+A^{-1}X)^{-1} = B^{-1}$$

$$(X+C)(D+A^{-1}X)^{-1} = B^{-1}A^{-1}B^{-1}$$

$$X + C = (B^{-1}A^{-1}B^{-1})(D+A^{-1}X)$$

$$= B^{-1}A^{-1}B^{-1}D + B^{-1}A^{-1}B^{-1}X$$

$$(I - B^{-1}A^{-1}B^{-1})X = B^{-1}A^{-1}B^{-1}D - C$$

$$X = (I - B^{-1}A^{-1}B^{-1})^{-1}B^{-1}A^{-1}B^{-1}D - (I - B^{-1}A^{-1}B^{-1})^{-1}C$$

$$X = BX$$

$$X - BX = 0$$

$$(I - B)X =$$

$$x - 3x = (1-3)x$$

$$(x+y)3 = 3(x+y)$$

$$Ix - BX$$

$\neq$

$AB = C$

$B = A^{-1}C \neq CA^{-1}$

$A_1 \quad A_2 \quad A_3$

matrix add.

$$\det(A_1 + A_2 + A_3)$$

$$= \det A_1 + \det A_2 + \det A_3$$

addition on $\mathbb{R}$