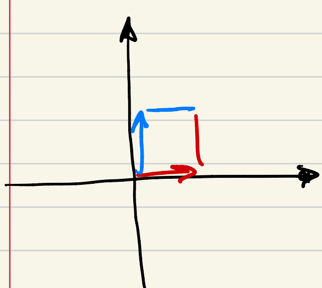
Lecture 12 : 07 | 20

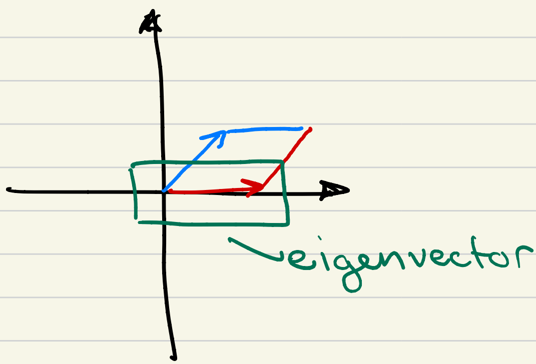
- Eigenvectors & Eigenvalues
  - Looking at square matrices
  - Linear transforms. $T : V \to V$
  - Consider V is $\mathbb{R}^n$ and all transforms are matricies

Eigen ~ "proper"

characteristic vector

Big idea: E.Vec are vectors whose direction remains unchanged under a linear transform.





Def: Given a L.T. $A \in M_{n \times n}(\mathbb{R})$, we want to find vectors v & eigenvalues λ such that

$$Av = \lambda v$$

Goal: find v & λ.

$$Av - \lambda v = 0$$

$$(A - \lambda I)v = 0$$

Recall

$Bv = 0$ & B is invertible, then

$$v = B^{-1} 0 = 0$$

if $A - \lambda I$ is invertible, we just get $v = 0$, doesn't help us.

So, we must find λ s.t. $A - \lambda I$ is not invertible.

- Require: $\det(A - \lambda I) = 0$

Ex: $\begin{pmatrix} -2 & 5 \\ 6 & -1 \end{pmatrix} = A$

$$A - \lambda I = \begin{pmatrix} -2-\lambda & 5 \\ 6 & -1-\lambda \end{pmatrix}$$

$$\boxed{\det(A - \lambda I) = (-2-\lambda)(-1-\lambda) - 30}$$

$$= (2+\lambda)(1+\lambda) - 30$$

characteristic polynomial.
(poly. in λ)

$$= 2 + 3\lambda + \lambda^2 - 30$$

$$= \lambda^2 + 3\lambda - 28$$

$$= (\lambda + 7)(\lambda - 4)$$

So $\lambda = -7$ or 4

$\lambda_1 = -7, \quad \lambda_2 = 4$

we want to find v_1, v_2

$$Av_1 = \lambda_1 v_1 , \quad Av_2 = \lambda_2 v_2$$

$$\begin{pmatrix} -2 & 5 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = -7 \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix}$$

$$-2v_1^1 + 5v_1^2 = -7v_1^1$$
$$6v_1^1 - v_1^2 = -7v_1^2$$

$$5v_1^1 + 5v_1^2 = 0$$
$$6v_1^1 + 6v_1^2 = 0$$

$$\boxed{v_1^1 + v_1^2 = 0}$$

$$v_1^1 = 1 \Rightarrow v_1^2 = -1$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 5 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$$

$$= -7 v_1$$

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 5 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} 4v_2^1 \\ 4v_2^2 \end{pmatrix}$$

$$-6v_2^1 + 5v_2^2 = 0$$

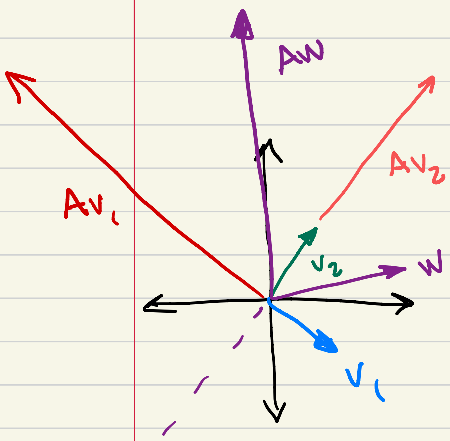
$$6v_2^1 - 5v_2^2 = 0$$

$$v_2^1 = \frac{5}{6} v_2^2 \qquad \begin{pmatrix} 5/6 \\ 1 \end{pmatrix}$$

$$1$$

$$v_2 = \begin{pmatrix} 5/6 \\ 1 \end{pmatrix}$$ v_1, v_2 are L.I.

also choose $v_2 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$



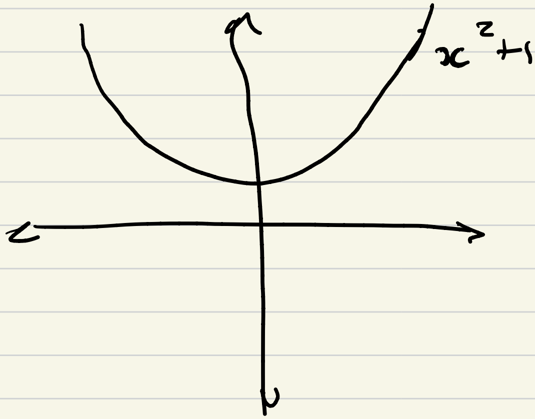
- Recall from precalculus:

  Fundamental theorem of algebra, which says every polynomial can be written as a product of terms of the form $(x-a)$, $a \in \mathbb{C}$

$\alpha_k \in \mathbb{C}$

$$\sum_{k=1}^{n} \alpha_k x^k = \prod_{k=1}^{n} (x - \beta_k)$$

$x^2 + 1$



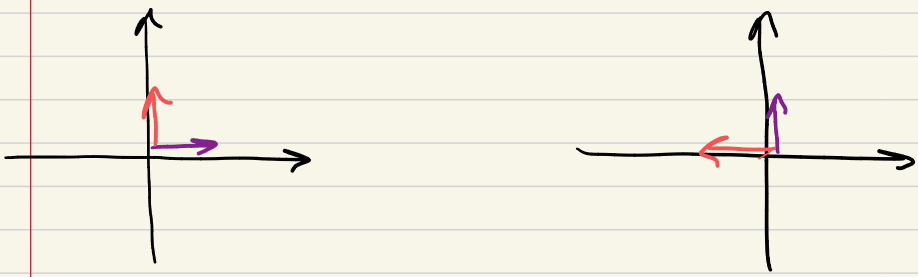
$(x+i)(x-i)$

$= x^2 + ix - ix - (i)^2$

$= x^2 + 1$

$$\boxed{\begin{aligned} ax^2 + bx + c &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}}$$

Quadratic Formula



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A$$

$$(A - \lambda I) = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 1$$

$$= \underline{(\lambda + i)}\,\underline{(\lambda - i)}$$

$$\lambda_1 = i, \quad \lambda_2 = -i$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = i\begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix}$$

$$-v_1^2 = i v_1^1 \Rightarrow \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

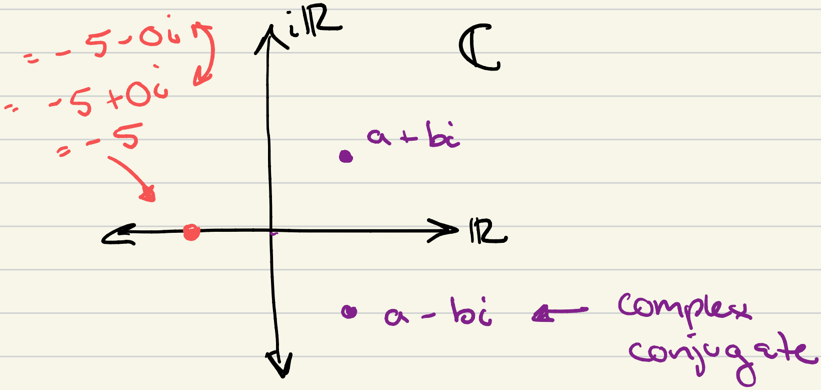
$$v_1^1 = i v_1^2$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix} = -i\begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix}$$

$$-v_2^2 = -i v_2^1 \quad = \quad v_2^2 = i v_2^1$$

$$v_2^1 = -i v_2^2 \qquad \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \leftarrow \text{L.I.}$$

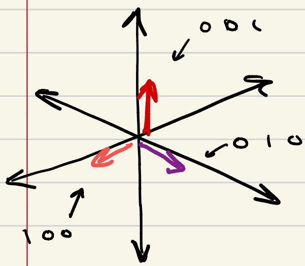


$3+2i$, c.c. is $3-2i$

$$(x + (3+2i))(x + (3-2i))$$

not: $(x + (3+2i))(x + (3-3i))$

never get



Fundamentally: a rotation defines a plane bc. we choose an axis of rotation

$$\begin{pmatrix} 0 & -2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = A$$

$$\det(A - \lambda I)$$

$$A - \lambda I = \begin{pmatrix} -\lambda & -2 & 0 \\ 3 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix}$$

$$-\lambda\big(-\lambda(-1-\lambda)\big) - (-2)\big(3(-1-\lambda)\big)$$

-1 -1 -1 -1 -1 -1

$$= -\lambda(\lambda(1+\lambda)) - 6(1+\lambda)$$

$\lambda + \lambda^2$

$$= -\lambda^2 - \lambda^3 - 6 - 6\lambda$$

$$= -(\lambda^3 + \lambda^2 + 6\lambda + 6)$$

$$\lambda_1 = -1, \quad \lambda_2 = -i\sqrt{6}, \quad \lambda_3 = i\sqrt{6}$$

Ex: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow (a, a), \ a \in \mathbb{R}$

$$(1, 1)$$

$$(1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1$$

$$\lambda(\lambda - 2)$$

$$\underline{\underline{\lambda_1 = 0}}, \quad \lambda_2 = 2$$

degenerate

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1^1 = -v_1^2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix} = \begin{pmatrix} 2v_2^1 \\ 2v_2^2 \end{pmatrix}$$

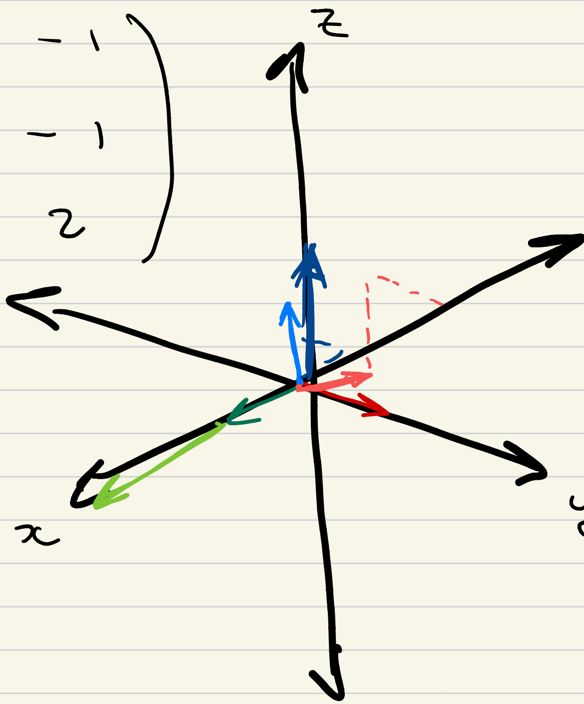
$$-V_2^1 + V_2^2 = 0$$

$$V_2^1 - V_2^2 = 0$$

$$\boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \downarrow & \downarrow & \downarrow \end{matrix}$$

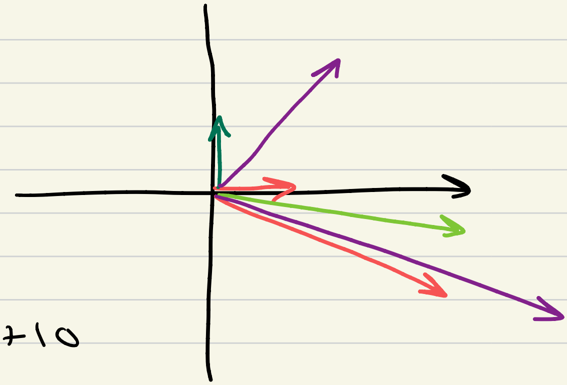
$$\begin{pmatrix} 3 & -4 & -1 \\ 0 & -1 & -1 \\ 0 & -4 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 3 & -4 & -1 \\ & \vdots & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ -4 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$\begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$$



$$(5-\lambda)(-1-\lambda) + 10$$

$$-5 - 4\lambda + \lambda^2 + 10$$

$$\lambda^2 - 4\lambda + 5$$

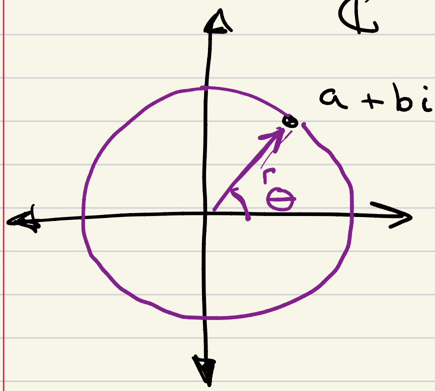
eigenvalues

$$\frac{4 \pm \sqrt{16 - 20}}{2} = \underline{\underline{2 \pm i}}$$

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$



$$re^{i\theta}$$

$$r = \sqrt{a^2+b^2}$$

$$= \sqrt{(a+bi)(a-bi)}$$

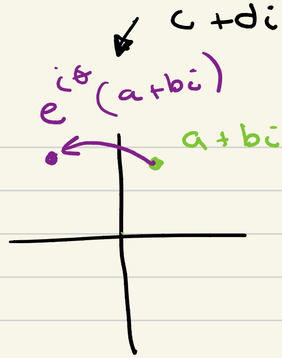
$$\sqrt{a^2+b^2}\, e^{i\theta} = \sqrt{a^2+b^2}\,(\cos\theta + i\sin\theta)$$

Rotation in $\mathbb{R}^2$ by θ

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\theta = \frac{\pi}{2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$e^{i\theta}$



$$(a+bi)e^{i\theta}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$$

$$\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\Updownarrow$$

$$\underbrace{\boxed{e^{i\theta}}(a+bi)} = c+di$$

$$e^{i\theta} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$v_k \in V \quad \dim V > 5$

$S = \{v_1, v_2, v_3, v_4, v_5\}$

$v_1 = v_2 + 2v_3 + 3v_4 + 6v_5 \leftarrow$ show by RREF

$\Rightarrow S$ is L.D.

it is <u>not</u> the case that $\swarrow$ must check L.I.

$\{v_1, v_2, v_3, v_4\}$ is L.I.

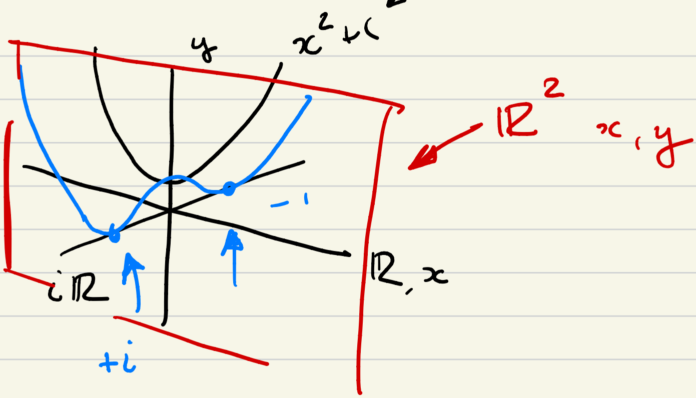
$v_1 = 2v_2 + 3v_3 - 2v_4$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \cancel{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}}, \cancel{\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}}, \cancel{\begin{pmatrix} 3 & 4 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}} \right\}$$

L.D.

$$\begin{pmatrix} 4 & 2 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$\mathbb{C}$ instead of $\mathbb{R}$

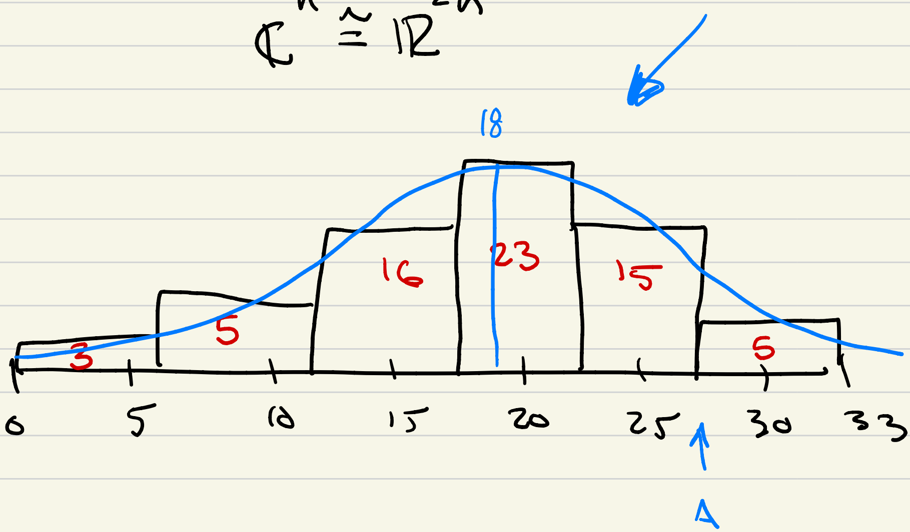


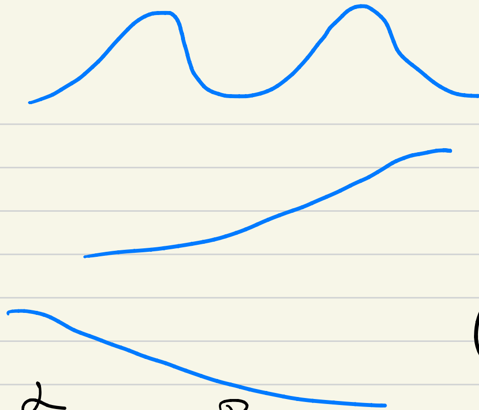
$$\mathbb{C}^2 \cong \mathbb{R}^4$$

$v_1, v_2 \in \mathbb{C}^2$ not $\mathbb{R}^2$

$$\mathbb{C}^2 \not\cong \mathbb{R}^3$$

$$\mathbb{C}^n \cong \mathbb{R}^{2n}$$





$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\underset{\alpha}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}, \quad \underset{\beta}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}, \quad \underset{\gamma}{\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}}$$

$a = 1$, $\quad c = -1$, $\quad b = -1$, $\quad 1$

$\alpha + \beta = 0 \qquad \alpha = -\beta$

$\alpha + \overset{1}{a}\gamma = 0 \qquad \underline{\gamma = -\alpha}$

$\overset{-\alpha}{\beta} + b\gamma = 0 \qquad -\alpha - b\alpha = 0$

$\Rightarrow b = -1$

$\overset{-\alpha}{\beta} + \gamma c = 0$

$-\alpha - \alpha c = 0$

$\Rightarrow c = -1$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

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