Lecture 13 : 07 / 22

- Eigenspace: Let λ_k be an eigenvalue of A w/ associated eigenvectors

  $v_k^1, \dots, v_k^j$ ← L.I.

  We call $E_k = \langle v_k^1, \dots, v_k^j \rangle$

  the k-th eigenspace. or the eigenspace of λ_k.

- Algebraic & geometric multiplicity of eigenvalues

  Alg. mult. : # of times the eigenvalue occurs

  Geo. mult. : dimension of E_k

$$\det(A - \lambda I) = (\lambda - 2)^{3} (\lambda - 1)$$

a.m. 1

↑ alg mult of 3

$$\begin{pmatrix} & & \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \\ v_1^4 \end{pmatrix} = 2\begin{pmatrix} v_1^1 \\ \vdots \\ v_1^4 \end{pmatrix}$$

↗ 3 deg of freedom for solving

$\Rightarrow$ generate 3 eigenvectors

1 deg of freedom

$\Rightarrow$ generate 1 eigenvector

geo. mult. $\leq$ alg. mult.

Ex: $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix} \Rightarrow -(\lambda-2)^2(\lambda-3)$

$\lambda_1 = 2, \ \lambda_2 = 2, \ \lambda_3 = 3$

$$\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

e.v.

$$a - b = 0$$
$$2b = 2b$$ same
$$-a + b = 0$$

$a + \alpha c$

$B(a+\alpha c)$

$a = b$

$$\begin{array}{c} a \\ b \\ \\ \end{array} \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix}$$

lets us choose 2 L.I. e.v.

$c = 0,\ c = 1$

$$a - b = 0$$
$$2b = 2b \Leftarrow \text{always true}$$

$$\begin{pmatrix} 1 \\ 1 \\ \alpha \end{pmatrix} = \boxed{\alpha b + (\alpha - 1) a}\quad \underset{a}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{b}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$ are L.I.

corresponding to 2

So both alg & geo mult for 2 are 2

Def: We say $A \in M_{m \times m}$ is nondefective if it has m L.I. eigenvectors.

(We say A is defective otherwise)

Prop: m distinct eigenvalues $\Rightarrow$ m L.I. eigenvec.

(homework problem)

$\Rightarrow$ if we have m distint eigenvalues, then A is nondefective.

Prop: A is nondefective if and only if alg mult = geo mult $\forall$ eigenspaces.

- Diagonalization

$\Lambda \quad \Lambda$

I,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ & 0 & & \lambda_m \end{pmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$$

$$M\Lambda = \Lambda M$$

Def: Let $A, B \in M_{m \times m}$, we say A is similar to B if $\exists$ an invertible matrix $P \in M_{m \times m}$ $\ni$.

$$B = P^{-1} A P$$

- Note Let $M = P^{-1}$, then

$$PBP^{-1} = A$$

$$= M^{-1} B M$$

So A similar to B means B similar to A

Ex: $A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 22 & 6 \\ -70 & -19 \end{pmatrix}$

$P = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}$

$$P^{-1} \qquad A$$

$$\begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \\ -13 & 7 \end{pmatrix}$$

$(P^{-1}A)$

$$\begin{pmatrix} 4 & -2 \\ -13 & 7 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 6 \\ -70 & -19 \end{pmatrix} = B$$

Thm: Similar matricies have the same eigenvalues.

* Two matricies w/ same eigenvalues may not be similar.

Pf: Let A be similar to B, then $\exists$ invertible P s.t. $B = P^{-1}AP$

$$\det(B - \lambda I) = \det(P^{-1}AP - \lambda \overbrace{P^{-1}P}^{I})$$

$$= \det(P^{-1}(AP - \lambda P))$$

$$= \det(P^{-1}(A - \lambda I)P)$$

$\det P^{-1} = \dfrac{1}{\det P}$

$$= \underline{\det(P^{-1})}\det(A - \lambda I)\underline{\det(P)}$$

$\det P \neq 0$

$$= \det(A - \lambda I)$$

$\Rightarrow$ A & B have the same characteristic eq

$\Rightarrow$ same eigenvalues! $\blacksquare$

- What is the nicest matrix w/ eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. possibly repeated

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$$

is pretty nice!

- Can we find P $\ni$

$$\boxed{P^{-1}AP} = \Lambda$$

want a similar diagonal matrix

Conjugation of P w/ A

- If so, when? If so, how?

Thm: $A \in M_{m \times m}$ is similar to a diagonal matrix if and only if A is nondefective.

We say A is diagonalizable if A is similar to a diagonal matrix.

$$\det(A - \lambda I) = 0$$

$$= \det(\Lambda - \lambda I) = \begin{pmatrix} \lambda_1 - \lambda & & & 0 \\ 0 & \lambda_2 - \lambda & & 0 \\ & & \ddots & \\ 0 & & & \lambda_m - \lambda \end{pmatrix}$$

$$\det(\Lambda - \lambda I) = \prod_{k=1}^{m} (\lambda_k - \lambda)$$

$\sum$ sum $\qquad \prod$ product

$$\sum_{k=1}^{m} \alpha_k = \alpha_1 + \alpha_2 + \cdots + \alpha_m$$

$$\prod_{k=1}^{m} \alpha_k = \alpha_1 \alpha_2 \cdots \alpha_m$$

- We can do better!

If A is nondefective, we get m L.I. eigenvectors $v_1, v_2, \ldots, v_m$

if we form the matrix

$$P = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

eigenvec. as columns

not unique

then $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$

αv_1

$$\begin{pmatrix} \alpha v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

Jordan Normal Form / J. Canonical F.

- Main idea: how close to a diagonal matrix can we get for a defective matrix.

Def: Given a matrix A, v is a generalized eigenvector of A if

$$(A-\lambda I)^k v = 0$$

for some $k \geq 1$, but

$$(A-\lambda I)^{k-1} v \neq 0.$$

- Note that for $k=1$, this is an eigenvector.

- gives a way to get more L.I. vectors for a defective matrix.

Ex: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = A$

$$\Rightarrow \quad (3-\lambda)(1-\lambda)^2$$

$$\lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = 1$$

$$V_1 = (1, 2, 2)$$

$Av = \lambda v$

$(A - \lambda I)v = 0$

both work up to yo

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$a = a$ → a can be anything

$b = 0$

$b + 2c = b \Rightarrow c = 0$

$3c = c \Rightarrow c = 0$

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$E_2 \quad \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ one dimensional

geo mult : 1

alg mult : 2

$\Rightarrow$ A is defective

$v \neq \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$$(A - I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \qquad (A-I)v \neq 0$$

WTS: $(A-I)^2 w = 0$

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2c = 0$

$4c = 0 \qquad c = 0 \qquad \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$4c = 0$

is a generalized E.v.

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = w_3$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} = P$$

$$P^{-1}AP = \begin{pmatrix} [3] & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Almost diagonal.

Def: A Jordan block corresponding to λ is a square matrix of the following form:

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & & & \\ 0 & \lambda & 1 & 0 & & \\ 0 & 0 & \lambda & 1 & & \\ & & & \ddots & 1 & \\ & & & & \lambda & 1 \\ & & & & 0 & \lambda \end{pmatrix}$$

1 along superdiag. $a_{i,i+1}$

λ along diagonal

* Note: We have 1×1 Jordan blocks which look like (λ), $[\lambda]$

Def: A matrix is in JNF if it is composed of Jordan blocks along its diagonal.

Ex:

$$\begin{pmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} & & & & \cdots 0 \\ & [8] & & & \vdots \\ & & \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix} & & \\ \vdots & & & [5] & \\ 0 \cdots & & & & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{pmatrix}$$

END OF 13

Quiz 2:

$$\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = (6-\lambda)(3-\lambda) + 2$$

$$= 18 - 15\lambda + \lambda^2 + 2$$

$$\lambda^2 - 9\lambda + 20$$

$$\frac{9 \pm \sqrt{81-80}}{2}$$

$$\frac{9 \pm 1}{2} = 4, 5$$

$$(\lambda - 5)(\lambda - 4)$$

$$\lambda_1 = 5, \quad \lambda_2 = 4$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a = b \quad \Rightarrow \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftarrow \text{eigen}$$

$$2a = 2b$$

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2a - b = 0 \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2a = b$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix}$$

Quiz 1 $\qquad 1, -3$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \qquad \det(A - \lambda I)$$

$$= \boxed{(1-\lambda)(-3-\lambda)}$$

$$\begin{pmatrix} a & \cancel{b} \\ \cancel{0} & c \end{pmatrix} \text{ or } \begin{pmatrix} a & \cancel{0} \\ \cancel{b} & c \end{pmatrix}$$

eigenvals. a, c

2 distinct e.v.
$\Rightarrow$ nondefective
$\Rightarrow$ similar to diag of e.v

$$\begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \text{ similar to } \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$(-1-\lambda)^2 - 4$$

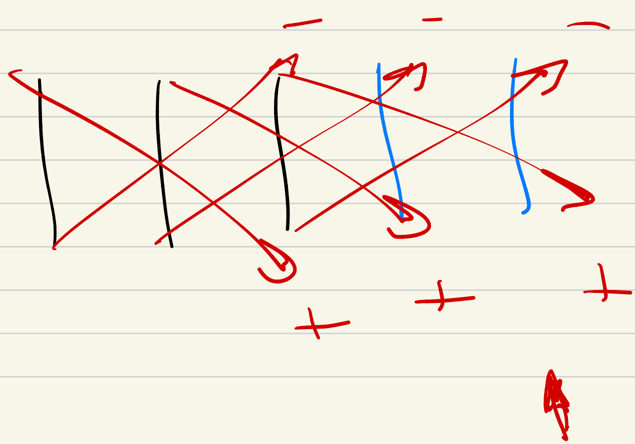
$$(\lambda+1)^2 - 4$$

$$\lambda^2 + 2\lambda + 1 - 4$$

$$= \lambda^2 + 2\lambda - 3$$

$$= \boxed{(\lambda+3)(\lambda-1)}$$

$$= (-3-\lambda)(1-\lambda)$$

2. 

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

m

$$a(ei - hf)$$
$$-b(di - fg)$$
$$+c(dh - ge)$$