Lecture 14: 07/24

- Recall that a Jordan Block looks like

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 & & & \\ 0 & \lambda & 1 & 0 & & & \\ 0 & 0 & \lambda & 1 & & & \\ & & & & \ddots & & \\ & & & & & \lambda & 1 \\ & & & & & 0 & \lambda \end{pmatrix}$$

- A matrix is in JNF if its main diagonal consists of Jordan blocks

$$\begin{pmatrix} 8 & 1 & & & & \\ 0 & 8 & & & & \\ & & 5 & & & \\ & & & 6 & 1 & \\ & & & 0 & 6 & \\ & & & & & 3 \end{pmatrix}$$

Question remains: how would we find the JNF?

$(A-\lambda I)^k v = 0, \quad k \geq 1$

Def: Let $A \in M_{m\times m}(\mathbb{R})$, v be a generalized eigenvector corresponding to λ. If p is the smallest integer $\ni$ $(A-\lambda I)^p v = 0$ then the ordered set

non-zero

order matters

$$\{ (A-\lambda I)^{p-1} v, (A-\lambda I)^{p-2} v, \ldots, (A-\lambda I) v, v \}$$

is called a cycle of generalized eigenvectors of length p.

- This cycle is always L.I.

- Two observations

① # of Jordan Blocks in JNF = # of L.I. eigenvectors of A

② Size of the Jordan Blocks = length of corresponding cycle

$\{ v_1, v_2, \ldots, v_j \}$

$$\underbrace{v_1, v_2}_{\text{e.v.}}, v_3, (A-\lambda I)v_3$$

v_3 ← g.e.v.

$A \in M_{4\times4}(\mathbb{R})$

$$(A-\lambda I)^2 v_3 = 0$$

2 JB of size 1

length 2

1 JB of size 2

$$\begin{pmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & 0 & \\ & 0 & \lambda_3 & 1 \\ & & 0 & \lambda_3 \end{pmatrix} = JNF(A)$$

$$(v_1 \quad v_2 \quad v_3 \quad (A-\lambda I)v_3)\ A$$

$$= JNF(A)$$

Ex: Let $A \in M_{4\times4}(\mathbb{R})$ w/ repeated eigenvalue λ, the the possible JNF looks like

① Diagonal

4 eigenvectors

$$\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$$

② 2, 2×2 JB

2 generalized e.v.

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

③ 2 1×1 JB, 1 2×2 JB

2 eigenvectors, one generalized eigenvector

$$\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

$$(A - \lambda I)^2 v = 0$$

$$\begin{pmatrix} \lambda & & & \\ & \lambda & 1 & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$$

⑤ 1 3×3, 1 1×1

$$\begin{pmatrix} \lambda & & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

(4) 4×4 JB

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \; 1 \end{pmatrix}$$

one generalized eigenvector

$$(A - \lambda I)^4 v = 0$$

Ex: Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$

find P $\ni$ $P^{-1} A P = \text{JNF}(A)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & \boxed{-1} & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Nullspace has dimension 1

$$\begin{matrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3 - \lambda \end{matrix}$$

$\det(A-\lambda I)$

$$= -\lambda\left[(-\lambda)(3-\lambda)+3\right] - 1(-1)$$

$$= -\lambda\left(\cancel{-3\lambda} + \lambda^2 + 3\right) + 1$$

$$= 3\lambda^2 - \lambda^3 - 3\lambda + 1$$

, still $\lambda = 1$, repeated e.v. of 1.

$$\mathrm{JNF}(A) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A-\lambda I)^2 v \neq 0$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ & \vdots & \\ & & \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a - 2b + c \neq 0$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v$$

$$(A - \ \ I)^2 v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - \ \ I) v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- <u>Every</u> $m \times m$ matrix is similar to a JNF matrix $P^{-1}AP$.

lower upper

- L-U factorization:

Goal: given an invertible matrix A, write A as the product of an upper & lower triangular matrix.

i.e. $A = LU$ find L find U

A = UL

- This factorization can be shown to be unique

$$AB$$

$$\underbrace{L_A U_A L_B U_B}$$ ① factor

② 3 mults.

$$L_A C U_B$$

- Upper triangular forms

→ All RREF are upper triangular but not all UT are RREF

Upper triangular form is row reduction to an upper triangular matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Leftrightarrow R_2 \leftrightarrow R_3$$

- Elementary Matrices

Performing row ops $\Leftrightarrow$ mult. by "elementary" matrices

Elementary matrix: any matrix obtained through row ops on I.

① Switching rows $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

② Multiply by a scalar $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$

③ add rows $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Let E be an elementary matrix, then

$EA \sim A$

Multiplication by elementary matrices gives the result of row ops.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\quad \overset{E_2}{R_1} - 2\overset{E_3}{R_2}$

$\boxed{-2R_2}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$

$R_1 = R_1 + R_2 \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

$-\frac{1}{2}R_2 \quad \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}}_{\text{Still Elementary}} A$$

$$= E_3 E_2 E_1 A$$

$$= E A$$

$$E_3 E_2 E_1 A = I$$

$$A^{-1} A = I$$

$$A E_3 E_2 E_1 = I$$

A is invertible if & only if $A \sim I$

- Suppose E_j is only type 2, 3

$E_k E_{k-1} \cdots E_1 A = U$ upper triangular

$$A = \underbrace{E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}}\, U$$

it turns out if $E_1, \ldots, E_k$ are only type 2 & 3, then this is lower triangular

$$L = E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$$

$$\boxed{A = LU}$$

LU Factorization / decomp

$E_1 I \sim I$

Ex:

$$A = \begin{pmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{pmatrix}$$

① $R_2 = R_2 - \frac{3}{2}R_1$

② $R_3 = R_3 + \frac{1}{2}R_1$

$$\sim \begin{pmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 9/2 & 5/2 \end{pmatrix}$$

③ $R_3 = R_3 + \frac{9}{13}R_2$

$$\sim$$

U

$$\begin{pmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{pmatrix}$$

2m row ops to get I

→ at most m row ops

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{9}{13} & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

$(E_3 E_2 E_1)^{-1}$

$E_1^{-1} E_2^{-1} E_3^{-1}$

* All elementary matrices are invertible

$$E_1^{-1} \qquad E_2^{-1}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{9}{13} & 1 \end{pmatrix} E_3^{-1}$$

$$E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 0 & 0 \\ -1/2 & -9/13 & 1 \end{pmatrix}$$

$$L$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 0 & 0 \\ -1/2 & -9/13 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{pmatrix}}_{U}$$

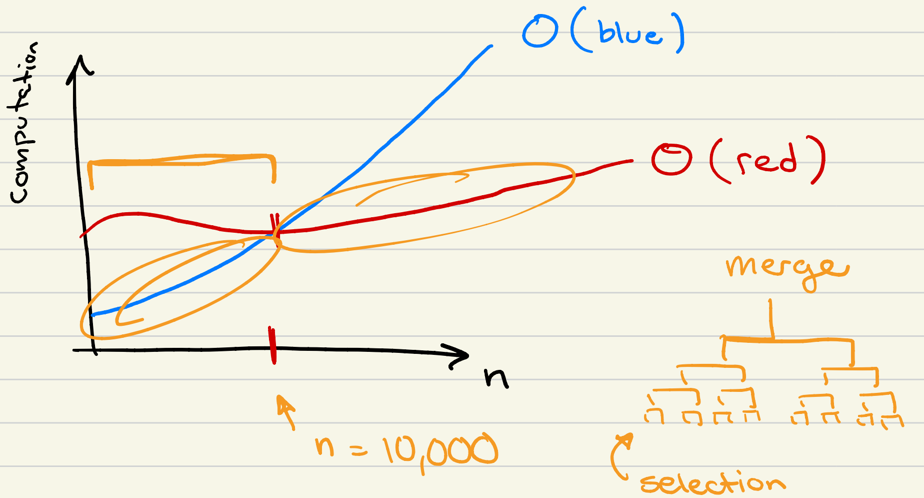
+2m for 2 determinant

AB

2m steps for factoring

m instead of m^2 computation
since
LU

END



$$\left(\begin{array}{c|c} A & I \end{array}\right) \sim \left(\begin{array}{c|c} I & \underset{B}{A^{-1}} \end{array}\right)$$

$$\boxed{\overline{a_{11}} \; 2a_{22}^2 + a_{12} - a_{21}} \neq 0$$

A invertible iff

B exists

$$A = \begin{pmatrix} 1 & 2 \\ 3 & b \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

$$\det(A) = b - 6$$

$$\underline{\underline{b \neq 6}}$$

$(1,0,0) \quad (0,1,0) \quad \cong \mathbb{R}^3 \quad (0,0,1)$

$$V = \left\langle \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \right\rangle$$

$A \in V \qquad V \subset \underbrace{M_{2\times 2}(\mathbb{R})}_{\text{dim } 4}$

$\langle (1,0,0) \rangle \cong \mathbb{R}$

$\mathbb{R}^m$

$$V = \langle \{ \boxed{(1,0,0)}, (0,1,0) \} \rangle$$

$\cong \mathbb{R}^2$

$\langle \{ (1,0), (0,1) \} \rangle$

$B = E_k \cdots E_1 A$

$$\det(B) = \prod_{j=1}^{k} \det(E_j) \det(A)$$

1

$$= \det(A)$$

$$I = \overbrace{E_k E_{k-1} \cdots E_1}^{A^{-1}} A$$

$$\underbrace{E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}}_{\text{any product}} = A$$

gives an invertible matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1 degree of freedom

$$x - y = 0$$

$$y - z = 0$$

$x = 1$

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle =$$

2D null space

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$