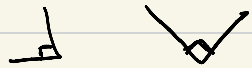
Lecture 15 : 07/27

- Orthoganality 

  Goal: generalize the notion of "angle" to a general vector space.

  For example in $\mathbb{R}^2$ 

  fairly clear what "angle" means in $\mathbb{R}^2$

  What does "angle" mean in $M_{n\times m}(\mathbb{R})$?

- Dot product: (scalar product)

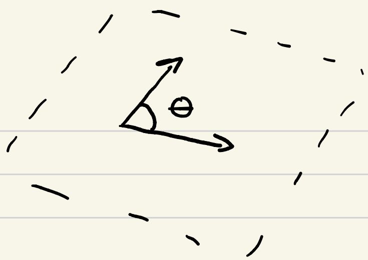
  $v, w \in \mathbb{R}^m$, $v = (v_1, v_2, \dots, v_m)$

  $$\langle v, w\rangle = v\cdot w = \sum_{k=1}^{m} v_k w_k$$

  in $\mathbb{R}^m$ this measures "angle" in the following sense:

  $$v\cdot w = |v||w|\cos\theta$$

  the angle between v & w in radians



Any two vectors lie on a plane

"Angle" is this θ

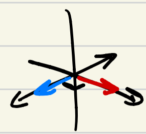
$$\theta = \cos^{-1}\left(\frac{v \cdot w}{|v||w|}\right)$$

$$|v| = \sqrt{\sum_{k=1}^{m} v_k^2} = \sqrt{v \cdot v}$$

Ex:

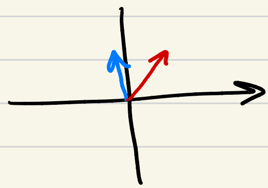
$$(0,1,0)\cdot(1,0,0)$$

$$= 0$$



$$(0,1)\cdot(1,1)$$

$$= 1$$



$$\theta = \cos^{-1}\left(\frac{(0,1)\cdot(1,1)}{1 \cdot \sqrt{2}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\boxed{= \pi/4}$$

$$(1,1,0)\cdot(1,1,0) = |(1,1,0)|^2$$

$$= 2 \quad \Rightarrow \quad |(1,1,0)| = \sqrt{2}$$

Aside: $x^2 = a, \quad a \geq 0$

$$x = \pm\sqrt{a}$$

$$|(1,1,0)|^2 = 2$$

$$\underline{|(1,1,0)|} = \pm\sqrt{2}$$

when talking about length, only take + root.

- Properties of dot product in $\mathbb{R}^m$

① $x \cdot x \geq 0$, why $x \cdot x = \sum_{k=1}^{m} \underbrace{x_k^2}$

② $x \cdot y = y \cdot x$, this follows from

$x_k y_k = y_k x_k$

$\alpha \in \mathbb{R}$

③ $\alpha(x \cdot y) = (\alpha x) \cdot y$

$$\alpha \sum_k x_k y_k = \sum (\alpha x_k) y_k$$

$$= \sum x_k (\alpha y_k)$$

④ $(x + y) \cdot z = x \cdot z + y \cdot z$

$$\sum_k (x_k + y_k) z_k$$

$$= \sum_k x_k z_k + \sum_k y_k z_k$$

For $\mathbb{R} = \mathbb{R}^1$

"multiplication"

① $x^2 \geq 0$, $xy = yx$, $\alpha(xy) = (\alpha x) y$

$(x+y) z = xz + yz$

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$

$f(x,y)$, $x \in V$, $y \in V$, $f(x,y) \in \mathbb{R}$

Def: Let V be a vector space. We say a mapping $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ is an inner product if it satisfies

1. $\langle x, x \rangle \geq 0$
2. $\langle x, y \rangle = \langle y, x \rangle$ commutative
3. $\alpha \langle x, y \rangle = \langle \alpha x, y \rangle$
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ distributes over $+$

$$\alpha \langle x, y \rangle \neq \langle \alpha x, \alpha y \rangle$$
$$= \alpha^2 \langle x, y \rangle$$

Ex: $C^0([a,b])$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

$$\theta = \cos^{-1}\left(\frac{\langle f, g \rangle}{|f||g|}\right)$$

$$|f| = \sqrt{\langle f, f \rangle}$$

$$f \cdot g = fg$$

$$f(x) \cdot g(x) = f(x) g(x)$$

1. $\langle f, f \rangle = \int_a^b f^2(x)\, dx \geq \min_{[a,b]} f^2 (b-a)$

$$\geq 0 (b-a) = 0$$

$\langle f, g \rangle = \underline{\underline{30}} \int_a^b f(x) g(x)\, dx$

(2) pretty clear

(3)

$$\alpha \int_a^b f(x)\, g(x)\, dx$$

$$= \int_a^b \big(\alpha f(x)\big)\, g(x)\, dx$$

(4) is also true

Def: A v.s. V w/ an associated inner product $\langle \cdot, \cdot \rangle$ is called an inner product space.

We define the <u>norm</u> on V, by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

generalizes notion of "length"

- Takeaway: Introduces a way to measure angles & length.
  V.S. don't have either!

V.S.

Ex: $A, B \in \mathcal{M}_{m \times m}(\mathbb{R})$

$$\langle A, B \rangle = \sum_{k=1}^{m} \sum_{j=1}^{m} a_{ij} b_{ij}$$

$$\left\langle \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \right\rangle$$

$$5 + 12 + 21 + 32 = 70$$

$$\| A \|_p^p = \sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj}^p, \quad p \geq 1$$

$$\| A \|_2 = \left(\sum_k \sum_j a_{kj}^2 \right)^{1/2}$$

$$\| A - A^* \|_2 \leftarrow \text{want small}$$

$$\| A - A^* \|_4$$

$$\left(\sum \sum a_{kj}^p \right)^{1/p}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

changing ruler
changes answer...

$$\|A\|_1 = 3 \qquad \|A\|_\infty = 2$$

$$\|A\|_2 = \sqrt{7}$$

$$\|A - A^*\|$$

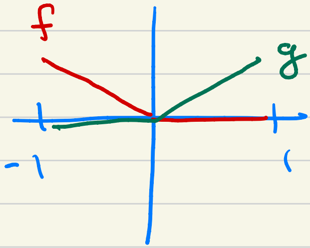
exact approximate

- Orthoganality

Let V be an i.p.s. $u, w \in V$

① If $\langle u, w \rangle = 0$, we say u, w are orthoganal

$$\int_a^b f(x) g(x)\, dx = 0$$



$$\int_a^b f g = 0$$

② A set $\{u_1, u_2, \ldots, u_m\}$ is orthoganal if $\langle u_i, u_j \rangle = 0 \ \forall\ i \neq j$

③ We say v is a unit vector if $\|v\| = 1$

④ We say a set is orthonormal if it is an orthogonal set of unit vectors

- Orthogonal and orthonormal bases
  - → These prove enormously useful

Thm: Any orthogonal set is L.I.

Pf: Let $\{v_1, \ldots, v_m\}$ be orthogonal assume $v_i \neq 0$ for any i

let $c_1, \ldots, c_m$ be scalars such that

$$\sum_{k=1}^{m} c_k v_k = 0$$

$\langle v, 0 \rangle = 0$

$\langle v, 0 \rangle = \langle v, 0 \cdot v \rangle$

$= 0 \cdot \langle v, v \rangle$

$= 0$

Fix any i, $1 \leq i \leq m$

$$0 = \langle v_i, \sum_k c_k v_k \rangle$$

$$= \sum_k c_k \langle v_i, v_k \rangle$$

since ortho. $\langle v_i, v_k \rangle = 0$ when $i \neq k$

$$= c_i \underbrace{\langle v_i, v_i \rangle}_{\neq 0}$$

$$\Rightarrow c_i = \frac{0}{\langle v_i, v_i \rangle} = 0$$

$c_i = 0 \quad \forall i, 1 \leq i \leq m.$

$\Rightarrow$ L.I.

$\blacksquare$

$$\sum_k c_k \langle v_i, v_k \rangle$$

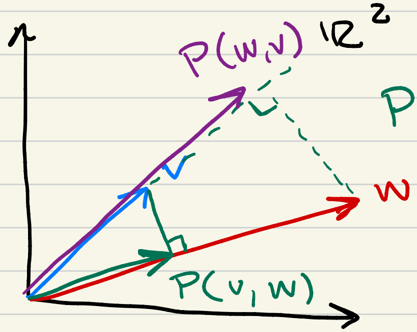
$= 0 \quad i \neq 1$ $\qquad \neq 0 \quad i = i$

$$= c_1 \langle v_i, v_1 \rangle + \cdots + c_i \langle v_i, v_i \rangle$$

$$+ \cdots + c_m \langle v_i, v_m \rangle \quad = 0 \quad i \neq m$$

$$= c_i \langle v_i, v_i \rangle$$

- Orthogonal projections



Projecting v onto w

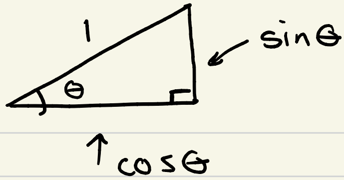
$P(v, w)$

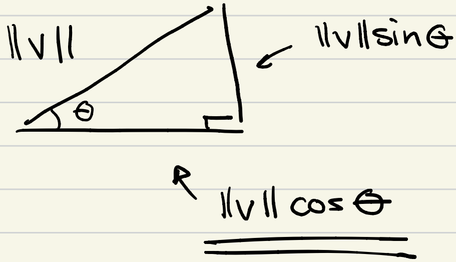
want a vector in the direction of w, with the length of the shadow of v.

$\frac{w}{\|w\|}$ a vector of length 1, in direction of w

$$\left\langle v, \frac{w}{\|w\|} \right\rangle = \|v\| \underbrace{\left\| \frac{w}{\|w\|} \right\|}_{=1} \cos\theta$$

$$= \|v\| \cos\theta$$





$$\boxed{P(v,w) = \frac{\langle v, w\rangle}{\|w\|^2} w}$$

$$= \left(\left\langle v, \frac{w}{\|w\|}\right\rangle\right) \frac{w}{\|w\|}$$

$$= \left(\frac{1}{\|w\|} \langle v, w\rangle\right) \frac{w}{\|w\|}$$

on $[-\pi, \pi]$

IBP

$$P(x^2, \sin x) = \boxed{\frac{\int_{-\pi}^{\pi} x^2 \sin x \, dx}{\int_{-\pi}^{\pi} \sin^2 x \, dx}} \sin x$$

$$\int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos(2x)) \, dx$$

$$= \alpha \sin x$$

- Write vectors in terms of orthogonal base

$$\{v_1, \dots, v_m\}$$

$$v = \sum_{k=1}^{m} \underbrace{\frac{\langle v, v_k \rangle}{\|v_k\|^2}}_{\alpha_k} v_k$$

for an orthonormal base, $\|v_k\| = 1$

$$v = \sum_{k=1}^{m} \langle v, v_k \rangle v_k$$

- $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$

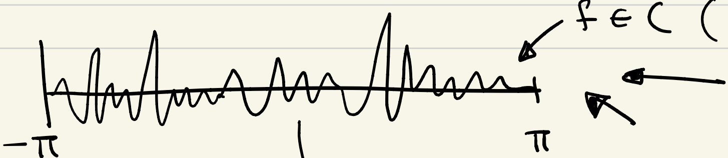
$[-\pi, \pi]$ $\qquad$ $C^0([-\pi, \pi])$

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$a_k = \langle f, \cos(kx) \rangle$$

$$b_k \leq \langle f, \sin(kx) \rangle$$

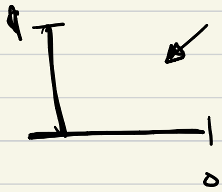
$f \in C^0([-\pi, \pi])$



$a_k\text{'s} = \{1, 3, -1, -2, 6, 10, 4, \tfrac{1}{2}\}$

$b_k = \{ \cdot \;\cdot\cdot \quad \cdot \;\cdot\cdot \}$

$C^0([0,1] \times [0,1])$



20,000

1,000 coefficents

an image at each pixel has a value (r, g, b)

0-255

$$f(a,b) = (r,g,b)$$

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$