MAT 022A Linear Algebra   Final   July 31, 2020

Answer all questions in a clear and concise manner. Label all work. Questions must be clearly labeled and in order. Exact answers only. No calculators. Show all work.
Each question must be on a separate sheet of paper, only turn in questions to be graded.
Citation rules from the syllabus apply, no collaboration. Open book, open internet (please cite sources, especially internet sources).
There are 10 problems, numbered 1-10. The test is scored out of 42 points.
Enjoy the rest of your Summer, you have been great students!

1. Prove the following statements
   (a) (1 point) If A is invertible, prove that A^k is invertible for any $k \geqslant 1$.
   (b) (1 point) Assuming A is invertible, prove that $\det((A^k)^{-1}) = \frac{1}{(\det(A))^k}$.
   (c) (1 point) Prove that $\det(\alpha A) = \alpha^m \det(A)$, $A \in \mathcal{M}_{m\times m}(\boldsymbol{R})$, $\alpha \in \boldsymbol{R}$, using the definition of the determinant (Hint: you may have seen this problem already in this course).
   (d) (1 point) Prove that if J is the Jordan normal form of A, that $\det(J) = \det(A)$.

2. (3 points) Find $A \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$ given that A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$ and eigenvectors $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

3. Given $f : \boldsymbol{R}^3 \to \boldsymbol{R}^4$,

$$f(x, y, z) = (x^2 + y^2, z, x + z, y^2 + z^3)$$

   (Hint: f is not linear)
   (a) (1 point) Find the kernel of f.
   (b) (1 point) Find the range of f.
   (c) (1 point) Is f is 1-1? Is f onto? Justify your answers.

4. (3 points) Let $f : \boldsymbol{R}^5 \to \boldsymbol{R}^5$ be given by

$$f(v, w, x, y, z) = (-v + w + x + y - z, w + x - y - z, v - w + x, w - x + y - z, v + w + x + y - z)$$

   Find the dimensionof the nullspace of f (Hint: Find a matrix representation of the transformation and then apply rank-nullity).

5. (a) (2 points) Describe the eigenvectors and eigenvalues of the linear transformation $\frac{d}{dx} : C^1(\boldsymbol{R}) \to C^0(\boldsymbol{R})$. Explain your answer.
   (b) (1 point) What do you notice about the number of eigenvalues that is different from the cases we have seen in class? Offer a short hypothesis as to why this is (will be graded lightly) (Hint: $C^1(\boldsymbol{R})$ is not finite dimensional).

6. (3 points) Prove that the space of functions with zero mean, that is f such that $\int_a^b f(x)\, dx = 0$, is a subspace of the vector space $\mathcal{R}([a, b])$, the space of Riemann integrable functions on $[a, b]$.

7. Decide whether or not the following situations are possible and justify your answers.
   (a) (1 point) $f : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ is onto.
   (b) (1 point) $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ is 1-1.

(c) (1 point) The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

is a basis of $\mathcal{M}_{3\times 2}(\boldsymbol{R})$?

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is linearly independent.

(e) (1 point) For a linear transformation $A : \boldsymbol{R}^m \to \boldsymbol{R}^d$ the dimension of the nullspace is larger than d.

(f) (1 points) Let $\Lambda \in \mathcal{M}_{4\times 4}$ be a diagonal matrix. Λ is similar to a matrix A which has eigenvalues $1, 2, 3$ with algebraic multiplicities $1, 2, 1$ and geometric multiplicities $1, 1, 1$ respectively.

8. Consider the ODE $\frac{d}{dx}y + y = 0$.

   (a) (3 points) Show that the solutions to this ODE form a subspace of the vector space $C^1(\boldsymbol{R})$.

   (b) (1 point) What is the smallest that the dimension of this subspace could be? Explain (Hint: e^{-x} is a solution to the ODE).

   (c) (1 point, HARDER) What is the largest that the dimension of this subspace could be? Explain.

9. We saw how JNF generalizes the notion of diagonalization, and we will now look at a similar concept which generalizes the notion of an inverse.

   The matrix **pseudo-inverse** of $A \in \mathcal{M}_{m\times m}(\boldsymbol{R})$ is the matrix $A^+ \in \mathcal{M}_{m\times m}(\boldsymbol{R})$ which satisfies the following four properties

   (a) $AA^+A = A$

   (b) $A^+AA^+ = A^+$

   (c) $(AA^+)^T = AA^+$

   (d) $(A^+A)^T = A^+A$

   Quickly convince yourself that this is indeed a generalization of the notion of A^{-1}.

   The following is true: Any matrix $A \in \mathcal{M}_{m\times m}(\boldsymbol{R})$ has a unique pesudo-inverse $A^+ \in \mathcal{M}_{m\times m}(\boldsymbol{R})$.

   (a) (3 points) Find the unique pseudo-inverse of $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (note that this matrix does not have an inverse).

   (b) (1 point) Prove that $(AB)^+ = B^+A^+$.

   (c) (1 point) Prove that $(A^+)^+ = A$.

   (d) (1 point) Prove that $A^+ = A^{-1}$ when A is invertible.

10. Given $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$,

    (a) (1 point) Classify this transformation as one of the basic types of transformations we discussed in class. Justify your answer.

    (b) (3 points) For what vectors w and scalars μ is the following true (ignore the trivial case of $w = 0$ or $\mu = 0$)

$$(A^2 - 2\mu A)w = -\mu^2 w$$

(c) (1 point) Write the matrix B whose eigenvectors are w and whose eigenvalues are μ (Hint: If you find yourself repeating problem 2 you're doing too much work).

(d) (1 point) Using parts (a) and (b), for what vectors w and scalars μ is the following true (ignore the trivial case of $w = 0$ or $\mu = 0$)

$$A(A^2 + 3\mu^2 I)w = \mu(3A^2 + \mu^2 I)w$$

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