Answer all questions in a clear and concise manner. Show all work. Submit one page per problem on Gradescope, properly select the subproblems and indicate if the solution requires multiple pages.
You may consult the internet, book, and peers. Cite any sources other than the book that you used. Citing Stack Exchange/Stack Overflow and/or Chegg is -.5 points per citation (subject to caveats discussed in lecture). If you are suspected of using an online resource without citation you will receive zero points for that problem.
If you work with peers, include the names of all in your group and do your own write-up **independently**, you should understand the solutions you are including in your write up. Copying on problems will result in zero points on that section.

1. The transpose of a matrix is obtained by "flipping" the matrix over it's diagonal. More precisely, given $A = (a_{ij})_{m\times n}$, we define (denoted :=) $A^T := (a_{ji})_{n\times m}$. For example

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 7 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad A^T = \begin{pmatrix} 1 & 5 \\ 3 & 7 \\ 4 & 1 \end{pmatrix}$$

Consider the space of 4×4 matrices $\mathcal{M}_{4\times 4}(\boldsymbol{R})$ and for $A, B \in \mathcal{M}_{4\times 4}(\boldsymbol{R})$ define $A \oplus B = A + B^T$.

Prove or disprove the following:

(a) (1 point) $\oplus$ is commutative on $\mathcal{M}_{4\times 4}(\boldsymbol{R})$.

(b) (1 point) $\oplus$ is associative on $\mathcal{M}_{4\times 4}(\boldsymbol{R})$ (you may use the fact that $(A+B)^T = A^T + B^T$ without proof).

(c) (1 point) What is $(-A)$?

2. Let $A, B \in \mathcal{M}_{n\times m}(\boldsymbol{R})$, $1 \leqslant n, m$

(a) (2 points) Prove that $(A+B)^T = A^T + B^T$.

(b) (1 point) Is $\mathcal{M}_{n\times m}(\boldsymbol{R})$ closed under $A \oplus B := (A+B)^T$? Explain.

3. We can define matrix multiplication as follows, let $A \in \mathcal{M}_{m\times n}(\boldsymbol{R})$ and $B \in \mathcal{M}_{n\times p}(\boldsymbol{R})$, then $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$ and

$$AB = A \times B := \left(\sum_{k=1} a_{ik} b_{kj} \right)_{m\times p}$$

(Notation: we say that "$f(A,B) = A \times B$ maps $\mathcal{M}_{m\times n}$ cross $\mathcal{M}_{n\times p}$ into $\mathcal{M}_{m\times p}$", since we can think of multiplication of a function taking two arguments from the Cartesian product. Symbolically we write

$$f : \underbrace{\mathcal{M}_{m\times n} \times \mathcal{M}_{n\times p}}_{\text{domain}} \to \underbrace{\mathcal{M}_{m\times p}}_{\text{range}}$$

Read Hammack section 1.2 for an introduction to Cartesian products.)

(a) (1 point) Show by example that for matrices $A, B \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$, $AB \neq BA$ in general.

(b) (2 points) Given $A, B \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$ show that $(AB)^T = B^T A^T$.

(c) (1 point) Show by example that $AA^T \neq A^T A$ for $A \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$.

(d) (1 point) Why is the result from (c) sufficient to show that $AA^T \neq A^T A$ in general for any $A \in \mathcal{M}_{n\times n}(\boldsymbol{C})$, $n \geqslant 2$.