MAT 022A Linear Algebra   Midterm   July 10, 2020

Answer all questions in a clear and concise manner. Label all work. Questions must be clearly labeled and in order. Exact answers only. No calculators. Show all work. Each question must be on a separate sheet of paper, only turn in questions to be graded. There are 8 problems, numbered 1-8. The test is scored out of 33 points.

1. (5 points) For what values of b does B^{-1} exist, given that

$$B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & b & 1 \\ b & 1 & 2b \end{pmatrix}$$

(Do not use a determinant).

2. (3 points) Let $C^1(\boldsymbol{R})$ denote the space of continuously differentiable functions on $\boldsymbol{R}$ (that is $f \in C^0(\boldsymbol{R})$ and we can differentiate f at all $x \in \boldsymbol{R}$, and $f' \in C^0(\boldsymbol{R})$). Prove or disprove that $C^1(\boldsymbol{R})$ is a vector space. (You may use the fact that if $f, g \in C^1(\boldsymbol{R})$, then αf and $f + g$ are in $C^1(\boldsymbol{R})$ for any $\alpha \in \boldsymbol{R}$)

3. Let $A, B \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$ and define $\oplus$ by

$$A \oplus B = \begin{pmatrix} a_{11} + b_{22} & a_{12} - b_{21} \\ a_{21} - b_{12} & a_{22} + b_{11} \end{pmatrix}$$

Answer the following and explain your answers.

   (a) (1 point) If a zero element exists under this operation what is it?
   (b) (1 point) If negatives exist what is the negative under this operation (I.e. given A, is there a B such that $A \oplus B = 0$)?
   (c) (1 point) Is $\oplus$ commutative?
   (d) (2 points) Is $\oplus$ associative?
   (e) (1 point) Does scalar multiplcation distribute over $\oplus$ (vector space rule 5)?

4. (2 points) A square matrix $A \in \mathcal{M}_{m\times m}(\boldsymbol{R})$ is called **lower triangular** if $a_{ij} = 0$ whenever $j < i$. For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ -5 & 8 & -3 \end{pmatrix} \in \mathcal{M}_{3\times 3}(\boldsymbol{R})$$

is upper triangular.

Given that $\mathcal{M}_{3\times 3}(\boldsymbol{R})$ is a vector space under addition and scalar multiplication, show that the space of lower triangular matrices is a subspace of $\mathcal{M}_{3\times 3}(\boldsymbol{R})$ (Hint: you don't need to prove all the properties).

5. (5 points) Determine whether the following set is linearly independent or linearly dependent. If the set is linearly dependent give a linearly independent subset and justify your answer.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -2 & 3 \end{pmatrix} \right\} \subset \mathcal{M}_{3\times 2}(\boldsymbol{R})$$

6. Given that

$$S = \{(1, 0, 1, 2), (0, 1, 0, 0), (0, 0, 1, 1)\}$$

(a) (2 points) Show the set is linearly independent.

(b) (2 points) Add a single vector to S to form a base of $\boldsymbol{R}^4$. Justify your answer.

7. (4 points) Find a base other than $\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}$ for the space $\mathcal{M}_{2\times 2}(\boldsymbol{R})$ and justify your answer.

8. Let $A \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$.

   (a) (2 points) Find the general form of $(A^T)^{-1}$ and provide a condition which allows us to take the inverse of the transpose.

   (b) (1 point) If A is invertible, does this mean A^T is invertible? Explain your answer.

   (c) (1 point) Provide a formula for $(A^T)^{-1}$ in terms of A, A^T and A^{-1}.

1. (5 points) For what values of b does B^{-1} exist, given that

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The End.