Midterm Solutions

1) $\begin{pmatrix} 1 & -1 & 1 \\ -1 & b & 1 \\ b & 1 & 2b \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & b & 1 & 0 & 1 & 0 \\ b & 1 & 2b & 0 & 0 & 1 \end{array}\right)$$

$$\overset{RREF}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2b^2-1}{b^2-3b-2} & \frac{2b+1}{b^2-3b-2} & \frac{b+1}{b^2-3b-2} \\ 0 & 1 & 0 & \frac{3b}{b^2-3b-2} & \frac{b}{b^2-3b-2} & \frac{2}{b^2-3b-2} \\ 0 & 0 & 1 & \frac{b^2+1}{b^2-3b-2} & \frac{b+1}{b^2-3b-2} & \frac{b-1}{b^2-3b-2} \end{array}\right)$$

So an inverse exists if

$$b^2 - 3b - 2 \neq 0$$

i.e. whenever

$$b \neq \frac{3}{2} \pm \frac{\sqrt{17}}{2}$$

2) Since we are given that C' is closed under + & scalar mult, it suffices to prove ① & ⑧.

We proved in HW2: P3 that C° a v.s. $C' \subset C^\circ$ and non-empty, since $0 \in C^\circ$ and $0' = 0 \in C^\circ \Rightarrow 0 \in C'$. By subspace thm and given info C' a v.s.

$\blacksquare$

3a) A zero element has $A \oplus O = A$

$$A \oplus B = \begin{pmatrix} a_{11} + b_{22} & a_{12} - b_{21} \\ a_{21} - b_{12} & a_{22} + b_{11} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow b_{ij} = 0$$

So $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero $\square$

Note $O \oplus A \neq A$ so cannot be a v.s.

3b) $A \oplus B = O \Rightarrow$

$$\begin{aligned} a_{11} + b_{22} &= 0 \\ a_{12} - b_{21} &= 0 \\ a_{21} - b_{12} &= 0 \\ a_{22} + b_{11} &= 0 \end{aligned} \Rightarrow \begin{aligned} b_{22} &= -a_{11} \\ b_{21} &= a_{12} \\ b_{12} &= a_{21} \\ b_{11} &= -a_{22} \end{aligned}$$

$$B = \begin{pmatrix} -a_{22} & a_{21} \\ a_{12} & -a_{11} \end{pmatrix}$$

$\square$

3c) No

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\neq$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\square$

3d)

4) Recall that if $W \subset V$, V a v.s. and W is closed under $+$ & $\times$, then W a v.s. (i.e. subspace). $\therefore$ it suffices to show closure:

(+) Let A, B be lower triang. 3x3

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & 0 & 0 \\ a_{21}+b_{21} & a_{22}+b_{22} & 0 \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

lower triangular

(x) $$\alpha \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & 0 & 0 \\ \alpha a_{21} & \alpha a_{22} & 0 \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{pmatrix}$$

lower triang.

5) $S = \left\{ \overset{S_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{S_2}{\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}}, \overset{S_3}{\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{S_4}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{S_5}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -2 & 3 \end{pmatrix}} \right\}$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

but this is not obvious, let

$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{R}$, we look at

$\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 + \alpha_4 S_4 + \alpha_5 S_5$

$$= \begin{pmatrix} \alpha_1 + \alpha_4 & 0 \\ \alpha_2 + \alpha_3 + \alpha_5 & \alpha_2 + \alpha_3 + \alpha_5 \\ -\alpha_2 + \alpha_4 - 2\alpha_5 & \alpha_1 + \alpha_2 + 3\alpha_5 \end{pmatrix}$$

Want to solve

$$\alpha_1 + \alpha_4 = 0$$
$$\alpha_2 + \alpha_3 + \alpha_5 = 0$$
$$-\alpha_2 + \alpha_4 - 2\alpha_5 = 0$$
$$\alpha_1 + \alpha_2 + 3\alpha_5 = 0$$

Alternativly, since $m < n$, the nullspace has dim 1+ and hence not L.I.

Show your row ops

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 3 & 0 \end{array}\right)$$

RREF

red is fixed by choice here

$$\sim \begin{pmatrix} \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ L.D.

$1 = 0$

Remove the last vector, then we have

$$\alpha_1 + \alpha_4 = 0$$

$$\alpha_2 + \alpha_3 = 0$$

$$-\alpha_2 + \alpha_4 = 0$$

$$\alpha_1 + \alpha_2 = 0$$

show your row ops!

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$\blacksquare$

$$\overset{RREF}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq I!$$

Remove another vector (s_4) to get

$$\begin{aligned} \alpha_1 \quad &= 0 \\ \alpha_2 + \alpha_3 &= 0 \\ -\alpha_2 \quad &= 0 \\ \alpha_1 + \alpha_2 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \overset{RREF}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ L.I.

Finally! $\blacksquare$

6a) We must show

$$\alpha(1,0,1,2) + \beta(0,1,0,0) + \gamma(0,0,1,1) = 0$$

$$\iff \alpha, \beta, \gamma = 0$$

i.e. solve

$$\begin{aligned} \alpha \quad\quad\quad\quad &= 0 \\ \beta \quad\quad\quad &= 0 \\ \alpha \quad\quad + \gamma &= 0 \\ 2\alpha \quad\quad + \gamma &= 0 \end{aligned}$$

1st gives $\alpha = 0$

2nd $\beta = 0$

4th / 3rd $\gamma = 0$

so set is L.I. $\square$

6b) We must add a vector so that for any $(a,b,c,d) \in \mathbb{R}^4$ $\exists$ $\alpha, \beta, \gamma, \lambda$ s.t.

$$(a,b,c,d) = \alpha(1,0,1,2) + \beta(0,1,0,0) + \gamma(0,0,1,1) + \lambda(m,n,p,q)$$

where we can choose m,n,p,q

(best to choose 1s & zeroes)

i.e. solve

$$\alpha + \lambda m = a$$

$$\beta + \lambda n = b$$

$$\alpha + \gamma + \lambda p = c$$

$$2\alpha + \gamma + \lambda q = d$$

choose zero to make $\alpha = a$, $\beta = b$

set to zero to see what happens ①

$\Rightarrow \gamma = c - a$

②

$$2a + c - a + \lambda q = d$$

$$\Rightarrow \lambda q = d - c - a$$

This is great, choose $\delta = 1$, then

$$\lambda = d - c - a$$
$$\gamma = c - a$$
$$\beta = b$$
$$\alpha = a$$

and we are done. So

$$\{(1,0,1,2),(0,1,0,0),(0,0,1,1),(0,0,0,1)\}$$

is a base of $\mathbb{R}^4$ $\blacksquare$

Note: we don't need to check new set is L.I. because if it were not, it would span same set as before, which isn't $\mathbb{R}^4$. (There are only 3 vectors).

7) Answers will vary. Need to find four matrices & show they form a base (i.e. LI & spanning)

Ex $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

$\blacksquare$

8a) The general form for 2×2 inverse

is $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\text{So} \quad (A^T)^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$\square$

8b) Yes, invertible if and only if

$ad-bc$ in both cases. $\square$

8c) $(A^T)^{-1} = (A^{-1})^T$

This question wasn't posed well

$\blacksquare$