Practice Final solutions:

2

$$\det(\alpha A)$$

$$= \sum_{\pi \in S_m} \sigma(\pi)(\alpha a_{1\pi_1}) \cdots (\alpha a_{m\pi_m})$$

$$= \alpha^m \sum_{\pi \in S_m} \sigma(\pi) a_{1\pi_1} \cdots a_{m\pi_m}$$

$$= \alpha^m \det(A)$$

$\blacksquare$

3a) $1 = \det(I)$

$= \det(AA^{-1})$

$= \det(A)\det(A^{-1})$

$= \det(A)\det(A^{T})$

$= (\det(A))^2 \Rightarrow \det(A) = \pm 1$ $\square$

3b) Elements of SL_m are "volume preserving" since their determinant has $\det A = c$

For example any rotation is in SL_m. also volume preserving shears.

We can't have reflections since in this case the determinant is negative.

No stretching since non unity det. $\square$

3c) Product of eigenvalues must be 1. $\blacksquare$

4) $AB = -BA$

$$\det(AB) = \det(-BA)$$
$$= (-1)^m \det(BA)$$
$$= -\det(B)\det(A) \quad m \text{ odd}$$
$$= -\det(AB)$$

So $\det(AB) = 0$ and either

$\det(A) = 0$ or $\det(B) = 0$

(or both) hence, at least

one isn't inv. $\blacksquare$

7) Given

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & & \\ 0 & a_{22}-\lambda & \cdots & & \\ \vdots & & & & \\ & & & & \\ 0 & \cdots & 0 & \cdots & a_{mm}-\lambda \end{pmatrix}$$

Cofactor expansion along first column shows

$$\det(A-\lambda I) = (a_{11}-\lambda)\underbrace{\begin{vmatrix} a_{22}-\lambda & a_{23} & \cdots \\ 0 & a_{33}-\lambda & \\ & \vdots & \\ 0 & & a_{mm}-\lambda \end{vmatrix}}_{\text{upper triangular}}$$

Repeat doing cofactor expansions to reach

$$\det(A - \lambda I) = \prod_{k=1}^{m} (a_{mm} - \lambda)$$

eigenvalues

So e.v. are $a_{11}, a_{22}, \dots, a_{mm}$ $\blacksquare$

8a) $\begin{pmatrix} \sqrt{\lambda_1} & 0 & & \\ 0 & \sqrt{\lambda_2} & & \\ & & \ddots & 0 \\ & & 0 & \sqrt{\lambda_m} \end{pmatrix}^2$

$$= \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}$$

$\square$

8b) We have $A = PDP^{-1}$

and $(P\sqrt{D}P^{-1})^2$

$$= P\sqrt{D}P^{-1}P\sqrt{D}P^{-1}$$

$$= P\sqrt{D}\,I\sqrt{D}P^{-1}$$

$$= P\sqrt{D}\sqrt{D}P^{-1}$$

$$= PDP^{-1} = A$$

so $P\sqrt{D}P^{-1}$ is a sq. root of A. $\square$

8c) Diagonalization (as in class) gives

$$\begin{pmatrix} 6 & -2 \\ -3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}}_{\text{"} = \text{"}\, D} \begin{pmatrix} 3/5 & 2/5 \\ -1/5 & 1/5 \end{pmatrix}$$

$$\Rightarrow \sqrt{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

so by (b)

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3/5 & 2/5 \\ -1/5 & 1/5 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 12 & -2 \\ -3 & 13 \end{pmatrix}$$

is a square root of A $\square$

8d) No, consider

$A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$, then $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is

a square root, but so is

$\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$

$\blacksquare$

9. Put J^T into JNF to get

$$J^T = S^{-1} M S$$

We must show $M = J$, since

$$J = I J I$$

is in JNF.

We know J, J^T have same eigenvalues w/ same algebraic multiplicities. (HW 5)

We must show that each λ_k has the same geometric multiplicity.

The geometric multiplicity is the dimension of the nullspace of $J - \lambda_k I$

Note that $(J - \lambda_k I)^T = J^T - \lambda_k I$
(check this)

So $J - \lambda_k I$ and $J^T - \lambda_k I$ have the same rank, and hence by rank-nullity they have the same dimension nullspace $\Rightarrow$ same geometric mult.

Same eigenvalues + same geo. mult $\Rightarrow$ same JNF $\blacksquare$

10) As in 9, must show A & A^T have same eigenvalues w/ same alg & geo mult.

Recall that $\det(A) = \det(A^T)$ and note that $(A - \lambda I)^T = A^T - \lambda I$ * check this!

So that $\det(A - \lambda I)$
$= \det((A - \lambda I)^T)$
$= \det(A^T - \lambda I)$

So A, A^T have same eigenvalues w/ same alg mult.

Recall that $\dim(N(A - \lambda I))$ = geo mult,

By row-column rank equiv (S-B: pg 151)

$\text{Rank}(A) = \text{Rank}(A^T)$ and by rank nullity

$\dim(N(A - \lambda_k I)) = \dim(N(A^T - \lambda_k I))$

and hence λ_k has same geo mult for A & A^T.

Finally, same eigenvalues w/ same alg & geo mult $\Rightarrow$ same JNF $\blacksquare$

Note that 9 is a special case of 10.

11) Only true for 2×2, your classmate Allie Baily presented the following counter-example for 3×3

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = 0$$

For 2×2, either A is diagonalizable or it isn't. If diagonalizable, then

$$A = P^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P$$

$$A^3 = P^{-1} \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix} P = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 0$$

$$\Rightarrow A = 0$$

If not, we get a Jordan Block

$$A = P^{-1} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} P$$

and

$$A^2 = P^{-1}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} P = 0$$

$$\Rightarrow \lambda = 0$$

$$\Rightarrow A^3 = 0$$

$\blacksquare$