Answer all questions in a clear and concise manner. Show all work. Submit one page per problem on Gradescope, properly select the subproblems and indicate if the solution requires multiple pages.
You may consult the internet, book, and peers. Cite any sources other than the book that you used. Citing Stack Exchange/Stack Overflow and/or Chegg is -.5 points per citation (subject to caveats discussed in lecture). If you are suspected of using an online resource without citation you will receive zero points for that problem.
If you work with peers, include the names of all in your group and do your own write-up **independently**, you should understand the solutions you are including in your write up. Copying on problems will result in zero points on that section.

1. Consider the subspace of $\boldsymbol{R}^3$ defined by

$$S = \{\, (x_1, x_2, x_3) \,|\, x_1 + 3x_2 - 5x_3 = 0 \,\}$$

Determine a spanning set for S.

2. Determine whether the following sets are linearly independent or dependent and explain.

   (a) $\{\sin x, \cos x\}$

   (b) $\{1, \sin^2 x, \cos^2 x\}$

   (c)

$$\left\{ \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \right\}$$

3. Given that $\mathcal{M}_{m\times m}(\boldsymbol{R})$ is a vector space, prove or disprove that S defined by

$$S = \left\{ \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \,\middle|\, a, b, c \in \boldsymbol{R} \right\}$$

is a subspace under matrix multiplication with scalar multiplication.

4. Define the operation $\oplus$ on $\mathcal{M}_{2\times 2}(\boldsymbol{R})$ by

$$A \oplus B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \oplus \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & a_{12}b_{22} \\ a_{21} + b_{12} - 3 & b_{21} - a_{22} + 1 \end{pmatrix}$$

   (a) If a zero element exists under this operation what is it?

   (b) If inverses exist what is the inverse under this operation?

   (c) Is $\oplus$ commutative?

   (d) Is $\oplus$ associative?

   (e) Does scalar addition distribute over $\oplus$ (vector space rule 5)?

5. Let $\mathbf{F}$ be the following subset of $\mathcal{M}_{2\times 2}(\boldsymbol{R})$

$$\mathbf{F} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a, b \in \boldsymbol{R} \right\}$$

which is indeed a field. Consider $\mathcal{M}_{2\times 2}(\boldsymbol{R})$ with normal addition and matrix multiplication between $\mathbf{F}$ and $\mathcal{M}_{2\times 2}(\boldsymbol{R})$.

(a) Does addition of scalars distribute over scalar multiplication, i.e. if $\alpha, \beta \in \mathbf{F}$ is it true that $(\alpha + \beta)u = \alpha u + \beta u$ (property 6).

(b) If an identity element exists, what does it look like (property 8)?

(c) Given $\alpha, \beta \in \mathbf{F}$ is it true that $(\alpha\beta)u = (\beta\alpha)u$ for all $u \in \mathcal{M}_{2\times 2}(\boldsymbol{R})$? Why or why not.

(d) Is $\mathcal{M}_{2\times 2}(\boldsymbol{R})$ with addition defined in the normal way and scalar multiplication defined by matrix multiplication with elements of the field a vector space?

6. If $(A^{-1})^2 = I$, is it true that $A^2 = I$?

7. Show that the space of n-th degree polynomials is a vector space. You may not use the fact that this space is contained in a larger vector space.

   This space consists of polynomials $\sum_{k=1}^{n} a_k x^k$.

The End.