Practice Midterm Solutions

1) Note that $\{(1,0,0),(0,1,0),(0,0,1)\}$ doesn't work (why?).

Instead, we note that

$$M = \{(-3,1,0),(5,0,1),(0,1,\tfrac{3}{5})\}$$

all solve the polynomial, and we also see $\langle M \rangle = S$ since linear combos will also solve.

$$\alpha(-3,1,0)+\beta(5,0,1)+\gamma(0,1,\tfrac{3}{5})$$

we have

$$-3\alpha+3\alpha+5\beta-5\beta+3\gamma-3\gamma=0$$

$\forall\ \alpha,\beta,\gamma \in \mathbb{R}$. $\Rightarrow \langle M \rangle \subset S$

Note that M is not L.I. ...
why?

Let $(a, b, c) \in S$, we must check

$(a, b, c) \in \langle M \rangle$

$$a = -3\alpha + 5\beta$$

$$b = \alpha + \gamma$$

$$c = \beta + \frac{3}{5}\gamma$$

$$\left(\begin{array}{ccc|c} -3 & 5 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 3/5 & c \end{array}\right)$$

RREF $\sim$ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ don't need bottom row!

$$\alpha(-3, 1, 0) + \beta(5, 0, 1) = (a, b, c)$$

$\Rightarrow \quad \beta = c \qquad a = -3\alpha + 5\beta$

$\alpha = b$

Check

$-3\alpha + 5\beta + 3\alpha - 5\beta = 0$ ✓

so $\alpha = b, \beta = c$ gives a sol. $\blacksquare$

2a) L.I. since $\sin x \neq \alpha \cos x$ for any α $\square$

2b) L.D. since $\sin^2 x = 1 - \cos^2 x$ $\square$

2c) Let

$$\{A, B, C, D\} = \left\{ \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \dots \right\}$$

then $\alpha A + \beta B + \gamma C + \delta D$

$$= \begin{pmatrix} \alpha + \beta + \gamma & -\beta - \gamma + 2\delta \\ 3\alpha - \gamma + \delta & -\alpha + \beta - \delta \end{pmatrix}$$

Goal: does this equal zero for non-zero $\alpha, \beta, \gamma,$ or δ?

$$\begin{aligned} \alpha + \beta + \gamma &= 0 \\ -\beta - \gamma + 2\delta &= 0 \\ 3\alpha - \gamma + \delta &= 0 \\ -\alpha + \beta - \delta &= 0 \end{aligned} \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 3 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 \end{array}\right)$$

$R_3 = R_3 - 3R_1$
$R_4 = R_4 + R_1$
$\sim$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & -3 & -4 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{array}\right)$$

$R_2 = -1R_2$
$R_1 = R_1 - R_2$
$R_3 = R_3 + 3R_2$
$R_4 = R_4 - 2R_2$
$\sim$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 7 & 0 \\ 0 & 0 & -1 & -5 & 0 \end{array}\right)$$

$R_3 = -R_3$
$R_2 = R_2 - R_3$
$R_4 = R_4 - R_3$
$\sim$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & -13 & 0 \end{array}\right)$$

$\cdots$

$$\sim \left(\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \end{array}\right)$$

so the set is L.I.

3) We use AB for A+B.

Since $M_{m\times m}$ a v.s., must show S closed under + and $\overset{\text{scalar}}{\times}$.

+ (mult)

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} ad & 0 \\ bd+ce & cf \end{pmatrix} \in S$$

so closed

$\times$: let $\alpha \in \mathbb{R}$

$$\alpha\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{pmatrix} \in S$$

So S a subspace. $\blacksquare$

4a) If a zero exists then

$A \oplus O = A$, i.e.

$$\begin{pmatrix} a_{11} - b_{11} & a_{12} b_{22} \\ a_{21} + b_{12} - 3 & b_{21} - a_{22} + 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$\Rightarrow b_{11} = 0, \; b_{22} = 1, \; b_{12} = 3$

$b_{21} = \underline{2a_{22} - 1}$

depends on A
So no zero element! $\square$

4b) $A \oplus (-A) = 0$

$$(-A) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} - b_{11} & a_{12} b_{22} \\ a_{21} + b_{12} - 3 & b_{21} - a_{22} + 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow b_{11} = a_{11}, \; b_{22} = 0, \; b_{12} = 3 - a_{21}$

$b_{21} = a_{22} - 1$

So $$\boxed{(-A) = \begin{pmatrix} a_{11} & 3 - a_{21} \\ a_{22} - 1 & 0 \end{pmatrix}}$$

$\square$

4c) Commutative

$$A \oplus B = \begin{pmatrix} a_{11} - b_{11} & a_{12} b_{22} \\ a_{21} + b_{12} - 3 & b_{21} - a_{22} + 1 \end{pmatrix}$$

$$\neq$$

$$B \oplus A = \begin{pmatrix} b_{11} - a_{11} & b_{12} a_{22} \\ b_{21} + a_{12} - 3 & a_{21} - b_{22} + 1 \end{pmatrix}$$

So not commutative. $\square$

4d) Associative

$$A \oplus B = \begin{pmatrix} a_{11} - b_{11} & a_{12} b_{22} \\ a_{21} + b_{12} - 3 & b_{21} - a_{22} + 1 \end{pmatrix}$$

$$B \oplus C = \begin{pmatrix} b_{11} - c_{11} & b_{12} c_{22} \\ b_{21} + c_{12} - 3 & c_{21} - b_{22} + 1 \end{pmatrix}$$

$$(A \oplus B) \oplus C = \begin{pmatrix} \underline{(a_{11} - b_{11}) - c_{11}} & (a_{12} b_{22}) c_{22} \\ (a_{21} + b_{12} - 3) + c_{12} - 3 & c_{21} - (b_{21} - a_{22} + 1) + 1 \end{pmatrix}$$

$$\neq$$

$$A \oplus (B \oplus C) = \begin{pmatrix} \underline{a_{11} - (b_{11} - c_{11})} \\ \\ \end{pmatrix}$$

So not associative! $\square$

4e) Does scalar mult distribute over $\oplus$?

That is $\alpha(A \oplus B) = (\alpha A) \oplus (\alpha B)$?

No, since

$$\alpha(a_{21} + b_{12} - 3) \neq \alpha a_{21} + \alpha b_{12} - 3$$

$\square$

5a) $$\alpha = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}$$

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\alpha + \beta = \begin{pmatrix} \alpha_1 + \beta_1 & -(\alpha_2 + \beta_2) \\ \alpha_2 + \beta_2 & \alpha_1 + \beta_1 \end{pmatrix}$$

$$(\alpha + \beta)u =$$

$$\begin{pmatrix} a(\alpha_1 + \beta_1) - c(\alpha_2 + \beta_2) & b(\alpha_1 + \beta_1) - d(\alpha_2 + \beta_2) \\ a(\alpha_2 + \beta_2) + c(\alpha_1 + \beta_1) & b(\alpha_2 + \beta_2) + d(\alpha_1 + \beta_1) \end{pmatrix}$$

$$\alpha u = \begin{pmatrix} a\alpha_1 - c\alpha_2 & b\alpha_1 - d\alpha_2 \\ a\alpha_2 + c\alpha_1 & b\alpha_2 + d\alpha_1 \end{pmatrix}$$

$$\beta u = \begin{pmatrix} a\beta_1 - c\beta_2 & b\beta_1 - d\beta_2 \\ a\beta_2 + c\beta_1 & b\beta_2 + d\beta_1 \end{pmatrix}$$

we see $\alpha u + \beta u = (\alpha + \beta)u$! $\square$

5b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity in $\mathbb{F}$

(check this!) $\square$

5c) It sufficies to check $\alpha\beta = \beta\alpha$

$$\alpha = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}$$

$$\alpha\beta = \begin{pmatrix} \alpha_1\beta_1 - \alpha_2\beta_2 & -\alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_2\beta_1 + \alpha_1\beta_2 & -\alpha_2\beta_2 + \alpha_1\beta_1 \end{pmatrix}$$

$$=$$

$$\beta\alpha = \begin{pmatrix} \beta_1\alpha_1 - \beta_2\alpha_2 & -\beta_1\alpha_2 - \beta_2\alpha_1 \\ \beta_2\alpha_1 + \beta_1\alpha_2 & -\beta_2\alpha_2 + \beta_1\alpha_1 \end{pmatrix}$$

so $\alpha\beta = \beta\alpha$ in general $\square$

d) It is, check all the properties, and ask on Discord if you run into an issue. $\blacksquare$

6) $A^{-1}A^{-1} = I$

$$AA^{-1}A^{-1} = AI = A$$

$$A^{-1} = A$$

$$AA^{-1} = A^2$$

$$I = A^2 \quad \checkmark$$

SO $(A^{-1})^2 = I \Rightarrow A^2 = I$ $\blacksquare$

7) Closed under $+$

$$\sum_{k=0}^{n} \alpha_k x^k + \sum_{k=0}^{n} \beta_k x^k$$

$$= \sum_{k=0}^{n} (\alpha_k + \beta_k) x^k \in P_n[x]$$

Under $\times$

$$\alpha \sum_{k=0}^{n} \alpha_k x^k$$

$$= \sum_{k=0}^{n} (\alpha \alpha_k) x^k \in P_n[x]$$

① $$\left(\sum_{k=0}^{n} \alpha_k x^k + \sum_{k=0}^{n} \beta_k x^k \right) + \sum_{k=0}^{n} \gamma_k x^k$$

$$= \sum_{k=0}^{n} (\alpha_k + \beta_k) x^k + \sum_{k} \gamma_k x^k$$

$$= \sum_{k=0}^{n} (\alpha_k + \beta_k + \gamma_k) x^k$$

$$= \sum_{k=0}^{n} \alpha_k x^k + \sum_{k=0}^{n} (\beta_k + \gamma_k) x^k$$

$$= \sum_{k=0}^{n} \alpha_k x^k + \left(\sum_{k=0}^{n} \beta_k x^k + \sum_{k=0}^{n} \gamma_k x^k \right)$$

(2) zero: $f(x) = 0 \in P_n[x]$

(3) $(-A) = \sum_k (-\alpha_k) x^k$

(4) Commutes since our $+$ is same as $+$ in $\mathbb{R}$.

(5) $\alpha \in \mathbb{F}$, $A, B \in P_n[x]$

$$\alpha(A+B) = \alpha \sum_k (\alpha_k + \beta_k) x^k$$

by scalar mult $\left\{ \begin{aligned} &= \sum_k (\alpha\alpha_k + \alpha\beta_k) x^k \\ &= \alpha A + \alpha B \end{aligned} \right.$

(6) $(\alpha + \beta) \sum_k \alpha_k x^k$

$$= \sum_k (\alpha+\beta) \alpha_k x^k$$

$$= \sum_k \alpha\alpha_k x^k + \beta\alpha_k x^k$$

$$= \alpha \sum_k \alpha_k x^k + \beta \sum_k \alpha_k x^k$$

(7) $$(\alpha\beta) \sum_k \alpha_k x^k$$

$$= \sum_k (\alpha\beta) \alpha_k x^k$$

$$= \sum_k \alpha (\beta \alpha_k) x^k$$ property of $\mathbb{R}$

$$= \alpha \left(\sum_k \beta \alpha_k x^k \right)$$

(8) $1 \in \mathbb{R}$ is the identity. $\blacksquare$