22A Review: 07/29

Topic Review

- Matrices
  - 1 → Represent systems of lin. eq.
  - → Add. & scalar mult.
  - 4 → Transpose
  - → Inverse
  - 3 → Matrix mult
  - 2 → Row operations and row equiv. matrices

    Row equiv. matrices correspond to equiv. sys. of eqs.
  - 3 ! → Gaussian elimination and RREF

- Vector spaces
  - 5 → vectors in $\mathbb{R}^d$, add. & scalar mult.
  - → Def. of a v.s. and some properties

    closed under + & x, satisfy (1) - (8)
  - ! → Subspaces and the subspace theorem

    know when, how, and why to apply
  - 6 → Linear combonations
  - → Span / spanning sets of a v.s.

    a spanning set is automatically a subspace!
  - → L.I. & L.D.

    know how to prove L.I. & L.D., understand what L.I. is, so if you forget procedure, you aren't stuck.
  - → Bases / Basis: A L.I. spanning set

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- → Represent any vector in a base
- ! → Dimension of a v.s.
  - → All bases have same # of elements
  - → Size of a L.I. set $\leq$ dim
  - → Size of a spanning set $\geq$ dim
  - → Dim of subspace
- → Four fundamental subspaces
  - ! → Columnspace
  - → Rowspace
  - ! → Nullspace — Know how dimension of nullspace affects range
  - → Left Nullspace
- → Solutions of a linear set of eqs. forms a vector space

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- → Rank of a matrix
- → Correspondance between dim N(A), free columns & free variables
- ! → Rank - Nullity theorem
- → Four possibilities for # of sols
- → Fundamental Thm. of L.A.

- The Determinant

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- → Volume function
- ! → Properties of the determinant
- → Permutations
  - → Transpositions & inversions
  - → Size of S_m and symmetric group
  - → pariety of a permutation $\sigma(p)$
- → Formal definition : $\sum_{\pi \in S_m} \sigma(\pi) a_{1\pi(1)} \cdots a_{m\pi(m)}$
- → Practical evaluation for 2×2 & 3×3.

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- → Minors
- → Cofactors
- → Method of cofactor expansion
- ! → Relation between the det. & invertibility

HW / Book → Various properties

$\det(AB) = \det(A)\det(B)$

$\det(A^T) = \det(A)$

• Linear Transformations

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- → Mappings
- ! → Linearity

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- → What simple linear transformations look like
- ! → kernel and range: They are subspaces
  - → kernel $= N(A)$
  - → range $= C(A)$
- → Composition of linear transformations
- ! → 1-1 and onto   Thm: relating dimension to 1-1 and onto
- ! → Linear transformations as matrices.

• Eigenvalues & Eigenvectors

- ! → Eigenvectors have fixed direction under a linear trans.: $Av = \lambda v$
- → How to find: $\det(A - \lambda I) = 0$ "characteristic polynomial"
- → Eigenvalues are roots of char. poly.

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! → Finding E.V. & E.v.

① compute & solve char. poly. for eigenvalues

② Plug in and look for eigenvectors

! → Complex eigenvalues may occur for real matrices. And correspond to rotation.

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→ Eigenspaces

→ Algebraic & Geometric multiplicity & their relations

→ Algebraic: # of times λ is a root

→ Geometric: dim of λ's eigenspace

! → Non-defective: m L.I. eigenvectors

→ Some theorems

• Diagonalization & JNF

→ Diagonal matrices: $\text{diag}(\lambda_1, \ldots, \lambda_m)$

→ Similarity

! → Similar matrices have same eigenvalues.

! → "Diagonalization Thm": A is similar to Λ (diag) if & only if A is nondefective

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→ How to compute

→ JNF for defective

→ Jordan blocks

→ Generalized eigenvectors

→ Def of JNF

→ How to find JNF

→ Cycles of GEV

! → Observations about JNF

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- ! → Every matrix has a JNF
- → Goal of LU : $A = LU$
- → upper triangular form
- → Elementary matrices
- → Comuting LU

- Inner Product Spaces
  - → Orthoganality $\langle x, y \rangle = 0$
  - → dot product on $\mathbb{R}^d$
  - → Properties of an inner product

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  - → Def. of an i.p.s.
  - → Orthogonal & orthonormal sets
    - → Any orthogonal set is L.I.
  - → Projections

You've got this!

- Prove set spans $\mathbb{R}^n$
- when to do JNF and why
- How to calculate span, L.I., base
- when is something a base
- Cofactor
- kernel & range
- Why iff is very powerful!
- finding eigenvec

- Vertical vs horizontal for L.I.
- 8b from MT
- 11 WW6
- 3a
- 7
- 5

$$A - \lambda I$$

WWG: 11

$$\begin{pmatrix} -1 & -1 & 0 \\ 9 & 6 & -1 \\ k & 0 & 0 \end{pmatrix} \longleftarrow$$

$$\begin{pmatrix} -1-\lambda & -1 & 0 \\ 9 & 6-\lambda & -1 \\ k & 0 & -\lambda \end{pmatrix}$$

$\det(A - \lambda I) =$

$$(-1-\lambda)\Big((6-\lambda)(-\lambda)\Big) + 1\Big(-9\lambda + k\Big)$$

$$= \lambda(1+\lambda)(6-\lambda) - 9\lambda + k$$

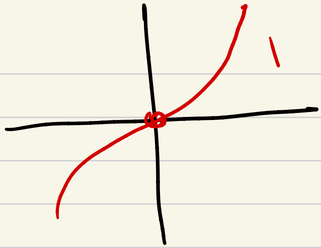
$$= (\lambda + \lambda^2)(6-\lambda) - 9\lambda + k$$

$$= 6\lambda - \lambda^2 + 6\lambda^2 - \lambda^3 - 9\lambda + k$$

$$-3\lambda + 5\lambda^2 - \lambda^3 + k = 0$$

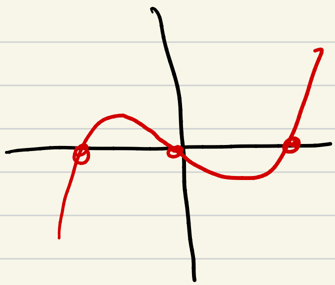
$$-\lambda(\lambda^2 - 5\lambda + 3) + k = 0$$

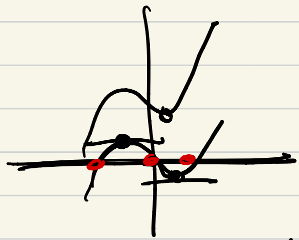
$$\frac{5 \pm \sqrt{25-12}}{2} = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$$



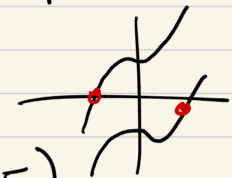
$-x^3 + 5x^2 - 3x$

$\boxed{-3x^2 + 10x - 3}$





$-x(x^2 - 5x + 3)$



$-(x^2 - 5x + 3) - x(2x - 5)$

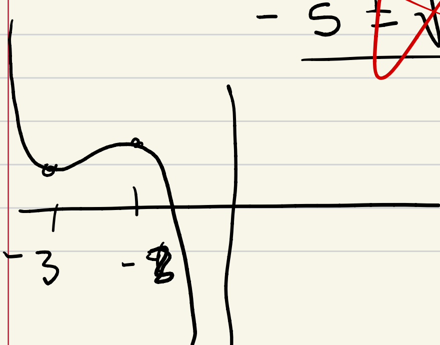
$-x^2 + 5x - 3 - 2x^2 + 5x$

$\boxed{-3x^2 + 5x + 2}$

$$\frac{-5 \pm \sqrt{25 - 24}}{2} = -\frac{5}{2} \pm \frac{1}{2}$$

$-\frac{4}{2}$ $\quad -2$

-3



$\lambda = -2 \quad , \quad \lambda = -3$

$-\lambda(\lambda^2 - 5\lambda + 3)$

$2(4 + 10 + 3) = 34$

$3(9 + 15 + 3)$

$3(27) = 81$

$-81 < k < -34$

4 $\alpha, \beta, \gamma, \delta \Rightarrow 6$ eqs

unique sol. for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \; 1 \\ 0 & 0 & 0 & 1 \; 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

no longer L.I.

$$\cong \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \\ \delta &= 0 \end{aligned}} \Rightarrow \text{L.I.}$$

$0 = 0$

$0 = 0$

$\alpha = 0$

$\beta = 0$

$\boxed{\gamma = \delta}$

$0 = 0$

$0 = 0$

$0 = 0$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$\dim(\langle A, B, C, D \rangle) = 4$

7) $\{A, B, C, D\}$

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha A + \beta B + \gamma C + \delta D$$

$$a = \alpha a_{11} + \beta b_{11} + \gamma c_{11} + \delta d_{11}$$

$$\vdots$$

$$d = \alpha a_{22} + \beta b_{22} + \gamma c_{22} + \delta d_{22}$$

chose 16 .?.

$A \sim I$

$$\underbrace{\begin{pmatrix} a_{11} & b_{11} & c_{11} & d_{11} \\ a_{12} & b_{12} & c_{12} & d_{12} \\ a_{21} & b_{21} & c_{21} & d_{21} \\ a_{22} & b_{22} & c_{22} & d_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}}_{g} = \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}_{\ell}$$

4 DOF

$Ag = \ell$ ← must show has at least one sol

$Ay = \ell$ has a unique solution

$y = A^{-1}\ell$ $\Rightarrow$ span & L.I.

$(a, b, c, d) = 0$

- 8b from MT:

$$(A^T)^{-1} = \frac{1}{\det A^T} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

iff $\det(A^T) \neq 0$

$\det(A^T) = \det(A) = ad - bc$

A^T invertible iff A invertible

so if A inv $\Rightarrow$ A^T inv

3a) $A \oplus O = A$, $\neq$ $O \oplus A = A$

$A \oplus B \neq B \oplus A$

$$\begin{vmatrix} 1 & -1 & -2 \\ 3 & 1 & 4 \\ 5 & 7 & 0 \end{vmatrix} = 5\begin{vmatrix} -1 & -2 \\ 1 & 4 \end{vmatrix} - 7\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}$$

- Cofactor Expansion

$$\begin{vmatrix} 1 & 2 & -1 & -2 \\ 3 & 0 & 1 & 4 \\ -3 & -1 & 0 & 1 \\ 5 & 0 & 7 & 0 \end{vmatrix}$$

$$\det(A) = +1\begin{vmatrix} 1 & -1 & -2 \\ 3 & 1 & 4 \\ 5 & 7 & 0 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 & 4 \\ -3 & 0 & 1 \\ 5 & 7 & 0 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

V.S.

$S \subset V$

- When is something a base? is S a base

* Maximal linearly independent set.

* if we add ANY vector to S, S will be L.D.

A base is L.I. and spans V

So given S, must check

(1) $\langle S \rangle = V$

(2) S is L.I.

$S = \{v_1, \ldots, v_n\}$

V a v.s. $S \subset V$

- How to calculate L.I., span

vector zero

S is L.I.? $\boxed{\sum_{k=1}^{n} \alpha_k v_k = 0}$ if and only if $\alpha_k = 0 \; \forall k$

depending on V, this generates some sys. of eqs.

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0 \qquad A \xrightarrow{\text{Row ops}} I \Rightarrow \text{invertible}$$

$\boxed{\text{if } A \text{ is invertible}}$, then S is L.I.

RREF also shows we have A^{-1}

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = A^{-1} 0 = 0$$

unique

$$\det(A) \neq 0 \Leftrightarrow S \text{ is L.I.}$$

$$\Rightarrow$$

$\det(A) = 0 \not\Rightarrow S$ is L.D.

For span we take arbitrary $w \in V$

and $\boxed{w = \sum_{k=1}^{n} \alpha_k v_k}$

$\rightarrow$ generates a system of EQs

$$B\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = b$$

if B is invertible, then

$\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$ has a unique solution

$\Rightarrow$ spanning set

infinitely many solutions $\Rightarrow$ spanning set

must show $B\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = b$ has at least one solution

For $\mathbb{R}^m$

$$(w_1, \dots, w_m) = \sum_{k=1}^{n} \alpha_k \underbrace{(v_1^k, \dots, v_m^k)}$$

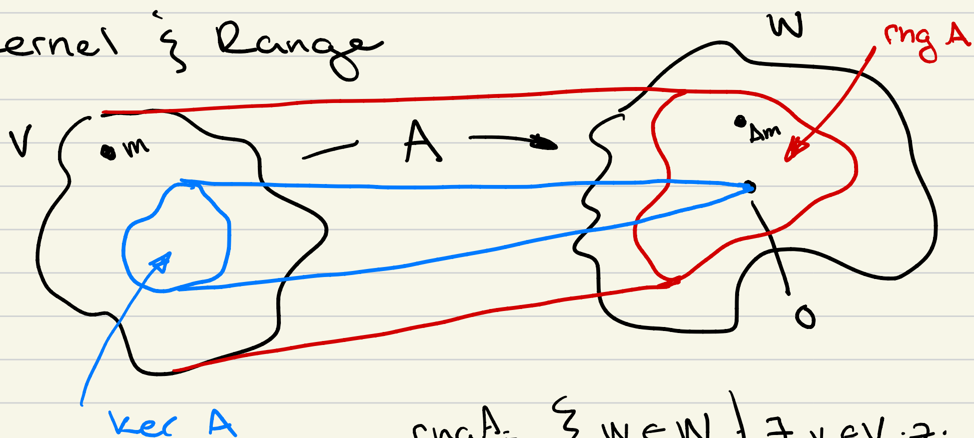
$$w_1 = \alpha_1 v_1^1 + \cdots \alpha_m v_1^m$$
$$\vdots \qquad\qquad \vdots$$
$$w_j = \alpha_1 v_j^1 + \cdots \alpha_m v_j^m$$
$$\vdots \qquad\qquad \vdots$$
$$w_m = \alpha_1 v_m^1 + \cdots + \alpha_m v_m^m$$

$$B = \begin{pmatrix} v_1^1 & v_1^2 & \cdots & v_1^m \\ v_2^1 & & & \vdots \\ \vdots & & & \vdots \\ v_m^1 & \cdots & & v_m^m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

$$= \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

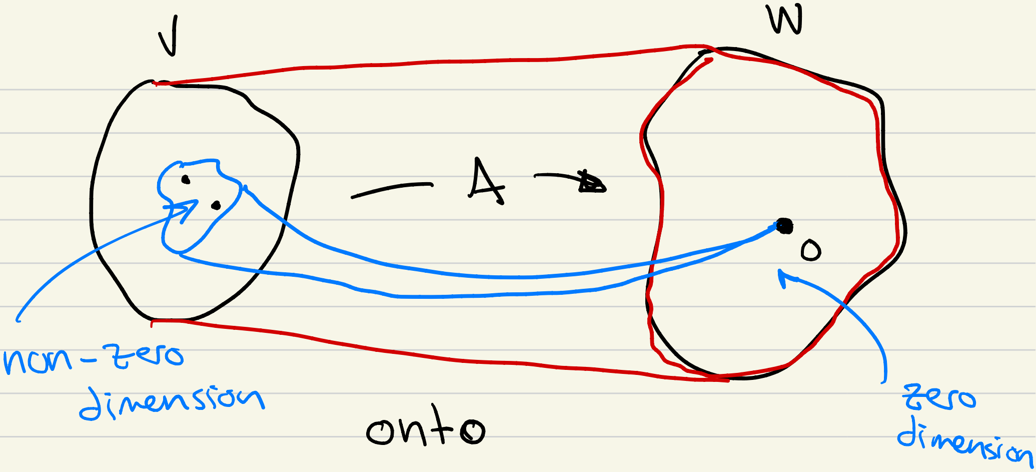
if has at least one sol, spans $\mathbb{R}^m$.

- Kernel & Range

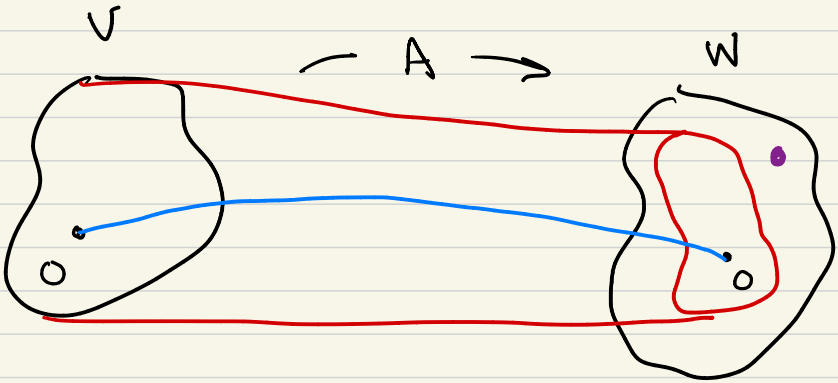


$\text{rng} A = \{ w \in W \mid \exists\, v \in V \text{ s.t. } Av = w \}$

$\ker A = \{ v \in V \mid Av = 0 \in W \}$



onto



1-1

- When to do JNF and why?

PF11: Prove there is no matrix A s.t. $A^2 \neq 0$ but $A^3 = 0$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = P^{-1} \Lambda P$$

$$= P^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P$$

$$A^3 = P^{-1} \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix} P = 0$$

$$\Rightarrow \boxed{\lambda_1^3 = 0, \ \lambda_2^3 = 0}$$

$$P A^3 P^{-1} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix} = 0$$

$$A^2 = P^{-1} \underbrace{\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}}_{0} P = 0$$

not diag.

$$A = P^{-1} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} P$$

$$\begin{pmatrix} \lambda^2 & \lambda+\lambda \\ 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$A^3 = P^{-1}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^3 P$$

$$P^{-1}\begin{pmatrix} \lambda^3 & 2\lambda^2 \\ 0 & \lambda^3 \end{pmatrix} P = 0$$

$$\Rightarrow \lambda = 0$$

$$A^2 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

No such matrix exists.

- Why if and only if is powerful

short answer : is iff means equivalent

W a sub. sp. iff W is a subset closed under + and ×

W a sub-sp $\Rightarrow$ W is a subset closed under + and ×

$\Leftarrow$

$$\underline{A \text{ inv.}} \iff \det(A) \neq 0$$

$\Rightarrow$

want to know easy, if A inv.

given A, all we do, is check det A

$\Longleftrightarrow$ $\qquad$ $\Rightarrow$

$\shortparallel$ $\qquad$ $A \Longleftrightarrow B$

$\Rightarrow$ $\qquad$ $A \Rightarrow B$

$\Leftarrow$ $\qquad$ $B \Rightarrow A$

- EV & EV

$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ want to find v, λ

$\cdot \exists \cdot$

$$Av = \lambda v$$

$$\Longleftrightarrow \quad Av - \lambda v = 0$$

$$\Longleftrightarrow \quad \underline{(A - \lambda I)} v = 0$$

$$\det(A - \lambda I) = 0$$

$v = (A - \lambda I)^{-1} 0$

$\Rightarrow$ $\underline{\underline{v = 0}}$

$$(1-\lambda)(3-\lambda) + 2$$

$$3 - 4\lambda + \lambda^2 + 2$$

$$\lambda^2 - 4\lambda + 5$$

$$\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\lambda_{\pm} = 2 \pm i$$

$$Av = \lambda_+ v_+ \qquad v_+ = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$Av - \lambda_+ v_+ = 0$$

$$\sim \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-(2+i) & 2 \\ -1 & 3-(2+i) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(1-(2+i))a = -2b$$

$$a = -\frac{2b}{-i-1} = \frac{2b}{i+1} = \frac{2b(i-1)}{-2}$$

$$= b(1-i)$$

$a = b(1-i)$, choose $b = 1$

$$v_+ = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-(2-i) & 2 \\ -1 & 3-(2-i) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a(-1+i) = -2b$$

$$a = \frac{2b}{1-i} = \frac{2b(-1-i)}{(1-i)(-1-i)}$$

$$= b(i+1)$$

again $b=1$

$$V_- = \begin{pmatrix} i+1 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{eigenvalues: } \lambda_+ = 2+i \\ \qquad\qquad\quad\; \lambda_- = 2-i \\ V_+ = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}, \quad V_- = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \end{array}}$$

$\lambda_1, \lambda_2 \cdot V_1, V_2$

$\mathbb{C}$ $\quad a \pm bi$

$\lambda_1 = a+bi$ $\qquad \lambda_+ = a+bi$

$\lambda_2 = a-bi$ $\qquad \lambda_- = a-bi$