PRELIMINARY EXAM IN ALGEBRA, SPRING 2019

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $G = C_4$ be the cyclic group of order 4. Give an example of a real representation M of G that is irreducible but such that $M \otimes_{\mathbb{R}} M$ is a reducible representation of G. (In other words, M is a simple $\mathbb{R}[G]$-module with $M \otimes_{\mathbb{R}} M$ not simple.)

2. Suppose that M is a $\mathbb{Q}[x]$-module such that $\dim_{\mathbb{Q}}(M) < \infty$. Show M is not projective .

3. Let R be a unital ring, M a left R-module. Is it true $\mathrm{Hom}_R(R,M) \simeq M$ as left R-modules? If so, prove it; if not explain why or give a counter example.

4. For which values of n is the ring $\mathrm{Mat}_{2,2}(\mathbb{Z}/n\mathbb{Z})$ isomorphic to a product of matrix rings over fields? Prove your answer.

5. Consider the additive group $\mathbb{Q}$ and its quotient group $G = \mathbb{Q}/\mathbb{Z}$. Prove that for every natural number n the group G contains a unique subgroup H of order n.

6. Determine if the group G given by the presentation

$$\langle x, y | xy = y^{2020} x \rangle$$

is nilpotent.

Preliminary Exam in Algebra
Spring 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. What is the splitting field of the polynomial $x^3+x+1 \in \mathbb{F}_5[x]$?

2. Prove that if G is a nontrivial nilpotent group, then its center $Z(G)$ is also nontrivial.

3. Let G be a group of order 50 and let n be the number of elements of order 5 in G. Find all possible values of n (and prove that the list is correct).

4. Consider the algebra $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Give a basis for A as a vector space over $\mathbb{R}$, and write out the product of every pair of basis vectors.

5. Give an example of a field F and a polynomial $f(x) \in F[x]$ which is irreducible but not separable. (Recall that a polynomial is *separable* if its has distinct roots in its splitting field.)

6. Calculate the character table (i.e., the table of the traces of complex irreducible representations) of the dihedral group D_4, by definition the finite group of order 8 with generators x, y and relations

$$x^4 = y^2 = 1 \qquad yxyx = 1.$$

Preliminary Exam in Algebra
Fall 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Prove that $\mathbb{Z}$ is a principal ideal domain.

2. Let $\mathbb{H}$ be the real quaternions. Then $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to which of the following rings? Prove your answer

(a) $\mathbb{C} \times \mathbb{C}$

(b) $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$

(c) $\mathbb{M}_2(\mathbb{C})$

(d) $\mathbb{M}_2(\mathbb{R})$

(e) $\mathbb{M}_2(\mathbb{H})$

(f) $\mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R})$

3. Let $f = x^4 - 14x^2 + 9 \in \mathbb{Q}[x]$. Compute the Galois group of f.

4. Solve the following:

(a) Prove that $R/I \otimes_R R/J \cong R/(I+J)$ for R a commutative ring, and $I, J \subset R$ are ideals.

(b) Find the dimension of $\mathbb{Q}[x,y]/(x^2+y^2) \otimes_{\mathbb{Q}[x,y]} \mathbb{Q}[x,y]/(x+y^3)$ as a vector space over $\mathbb{Q}$, or explain why it is infinite.

5. Solve the following questions:

(a) If F is a field, prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over F.

(b) Show that $f(x) = x^2 + 2x + 2$ is irreducible in $\mathbb{Q}[x]$, and find the inverse of $1 + x$ in $\mathbb{Q}[x]/(f(x))$.

6. Let V be the subspace of $\mathbb{C}^3$ spanned by $v_1, v_2 = (1,-1,0), (0,1,-1)$, which is an invariant subspace under the permutation action of S_3, and so gives a two-dimensional representation $\rho : S_3 \to GL(V)$.

(a) Write down the matrices of $\rho(\sigma)$ in this basis.

(b) Describe a Hermitian inner product $\langle , \rangle$ on V in the basis v_1, v_2, which is G-invariant, i.e. $\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$.

(c) Describe the tensor product $V \otimes_R V$, as an R-module where $R = \mathbb{C}[H]$ and $H \subset S_3$ is the subgroup $\{1, (123), (132)\}$.

Preliminary Exam in Algebra

Spring, 2017

Instructions:

(1) All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
(2) Use separate sheets for the solution of each problem.

Problem 1. Let G be a finite group and $H \subseteq G$ a subgroup such that $[G : H] = p$ where p is the smallest prime dividing $|G|$. Show that H is a normal subgroup of G.

Problem 2. Let k be a field, and let $f \in k[x]$ be of degree $= n \geq 1$. Let K be the splitting field of f (over k, embedded in some fixed algebraic closure of k). Prove that $[K : k] \leq n!$.

Problem 3. Show that the free group of rank 2 is not solvable.

Problem 4. Give an example of a projective R-module that is not free for $R = \mathbb{R}[x]/(x^4 + x^2)$.

Problem 5. Let G be the nonabelian group of order 57.
(a) How many 1-dimensional characters does G have ?
(b) What are the dimensions (aka degrees) of the other irreducible characters of G ?

Problem 6.
Let $\mathbb{F}$ be a finite field.
(a) Show that $|\mathbb{F}| = p^r$ for some prime p.
(b) Show that the multiplicative group $\mathbb{F}\backslash\{0\}$ is a cyclic group.

PRELIMINARY EXAM IN ALGEBRA
FALL 2017

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Suppose that $F \subseteq K$ is an inclusion of fields and let $\alpha, \beta \in K$ be two elements which are algebraic over F. Show that $\alpha + \beta$ is also algebraic over F.

2. Let $f \in \mathbb{Q}[x]$ be the minimal polynomial of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$, and let K be the splitting field for f over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$ and what is $\mathrm{Gal}(K/\mathbb{Q})$? (Note that you are not required to find f.)

3. Let $M_n(\mathbb{R})$ denote the ring of $n \times n$ matrices over $\mathbb{R}$, and consider a (possibly non-unital) ring homomorphism $f : M_{n+1}(\mathbb{R}) \to M_n(\mathbb{R})$. Can f be non-zero?

4. Find all maximal ideals of the ring $\mathbb{F}_7[x]/(x^2+1)$ and the ring $\mathbb{F}_7[x]/(x^3+1)$.

5. Let F be a field and let $p, q \in F[x]$ be polynomials over F. Show that

$$F[x]/(p) \otimes_{F[x]} F[x]/(q) \cong F[x]/(\gcd(p,q))$$

as $F[x]$-modules.

6. Prove that if p is a prime number, then every group G with p^2 elements is abelian.

Preliminary Exam in Algebra

Spring, 2016

Instructions:

(1) All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
(2) Use separate sheets for the solution of each problem.

Problem 1. Let $\mathbf{k}$ be a field and let $R = Mat_{n,n}(\mathbf{k})$ be the ring of $n \times n$ matrices with entries from $\mathbf{k}$. Let $f : R \to S$ be any ring homomorphism. Show that f is either injective or zero.

Problem 2. Let R be a ring with the identity, consisting of p^2 elements. Show that R is commutative.

Problem 3. Let G be a group generated by elements a, b each of which has order 2. Prove that G contains a subgroup of index 2.

Problem 4. Prove that every finite group G of order > 2 has a nontrivial automorphism.

Problem 5. Find all possible Jordan canonical forms for a matrix $A = T((123))$ if T is a two dimensional complex linear representation of the symmetric group S_3.

Problem 6. Find the smallest nonnegative integer $C \geq 0$ for which $R_c = Z[x]/(c, x^2 - 2)$ is
(a) a domain
(b) a field.

PRELIMINARY EXAM IN ALGEBRA
FALL, 2016

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let F_n be the free group on n generators with $n \geq 2$. Prove that the center $Z(F)$ of F is trivial.

2. Let G be a finite group that acts transitively on a set X of cardinality ≥ 2. Show that there exists an element $g \in G$ which acts on X without any fixed points. Is the same true if G is infinite?

3. Show that every linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ has both a 1-dimensional invariant subspace and a 2-dimensional invariant subspace.

4. Let $I, J \subseteq R$ be ideals in a principal ideal domain R. Prove that $I + J = R$ if and only if $IJ = I \cap J$.

5. Let F be a finite field and let L be the subfield of F generated by elements of the form x^3 for all $x \in F$. Prove that if $L \neq F$, then F has exactly 4 elements.

6. Show that the $\mathbb{R}$-modules $L = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $M = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic.

Algebra Prelim Problems
Spring, 2015

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1.

Show that if M is a nondiagonalizable complex matrix and M^n is diagonalizable then $\det(M) = 0$.

Problem 2.

Find the degree of the field extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$.

Problem 3.

Show that if G is an infinite simple group then every proper subgroup has infinitely many conjugates. Use this to conclude that G has an infinite automorphism group.

Problem 4.

Find a quotient ring of $Z[x]$ which is a principal ideal domain but not a field.

Problem 5.

Let $R = \mathbb{Q}[X]/(X^3 - 2)$.

(a) Is R a field? Explain.

(b) Run the extended Euclidean algorithm on $X^3 - 2$ and $X^2 - X + 1$ to find polynomials $A(x)$ and $B(x)$ with

$$A(X)(X^3 - 2) + B(X)(X^2 - X + 1) = gcd(X^3 - 2, X^2 - X + 1).$$

(c) Does $[X^2 - X + 1]$ have a multiplicative inverse in R? If yes, find it.

Problem 6.

Let G be a finite group and $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ a representation.

(a) Show: $\delta\colon G \to \mathbb{C}$, $g \mapsto det(\rho(g))$ is a linear character of G (i.e. a group homomorphism to the multiplicative group).

(b) Show: If $\delta(g) = -1$ for some $g \in G$, then G has a normal subgroup of index 2.

(c) Show: If G has order $2k$, k odd, then G has a normal subgroup of index 2.

(d) Let $\chi(g) = tr(\rho(r))$ and $g \in G$ an involution. Show: (i) $\chi(g)$ is an integer; (ii) $\chi(g) \equiv \chi(1)$ mod 2; (iii) if G has no normal subgroup of index 2, then $\chi(g) \equiv \chi(1)$ mod 4.

Preliminary Exam in Algebra
Fall, 2015

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1.
Let G be a finite group such that all Sylow subgroups of G are normal and abelian. Show that G is abelian.

Problem 2.
For a finite group G define the subset $G^2 = \{g^2 : g \in G\} \subset G$. Is it true that G^2 is always a subgroup?

Problem 3.
What is the smallest possible n for which there is an n by n real matrix M which has both:
(a) the rank of M^2 is smaller than the rank of M,
(b) M leaves infinitely many length one vectors fixed.

Problem 4.
Let I denote the ideal in the ring $\mathbb{Z}[x]$ generated by 5 and $x^3 + x + 1$. Is I a prime ideal?

Problem 5.
Show that two free groups are isomorphic if and only if they have equal ranks.

Problem 6.
Find the $\mathbb{Q}$-dimension of the splitting field over $\mathbb{Q}$ of $x^5 - 3$.

Spring 2014: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Find the smallest order of a group which is not cyclic and not isomorphic to a subgroup of S_5 (the symmetric group on five objects).

Problem 2: Consider the four dimensional real vector space

$$V = \{f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{R}\}$$

with the $\mathbb{R}[x]$-module structure given by shifting so that $(xf)(r) = f(r+1)$. Find a direct sum decomposition of V into irreducible $\mathbb{R}[x]$-modules.

Problem 3: Let p, q be prime numbers with $p < q$ such that p is not a divisor of $q - 1$. Let G be a group of order qp. Which of the following is true: (a) G is always simple. (b) G is never simple. (c) G could be simple or non-simple.

Problem 4: Give an example of a ring R and an R-module M which is projective but not free.

Problem 5: Let V be a linear representation of a group G over the field $\mathbb{Q}(\sqrt{2})$, and let $\chi : G \to \mathbb{Q}(\sqrt{2})$ be its character. Then V is also a vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$, and is again a linear representation of G. Express its character $\chi_{\mathbb{Q}}$ in terms of the original character χ.

Problem 6: Let $\mathbb{C}(x)$ be the field of complex rational functions, i.e., the fraction field of the polynomial ring $\mathbb{C}[x]$. Let $\mathbb{C}(y)$ be another copy of the same field in the variable y. This field is an algebra over $\mathbb{C}$, hence $\mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ is another algebra over $\mathbb{C}$. Is it also a field?

Algebra Prelim Problems
Fall, 2014

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1.

Let G_1, G_2 be finite index subgroups of a group G. Show that the intersection $G_1 \cap G_2$ also has finite index in G.

Problem 2.

Let G be a finite group and $N \subseteq G$ be a subgroup of index p, where p is the smallest prime dividing $|G|$. Prove N is a normal subgroup of G ($N \triangleleft G$).

Problem 3.

Does the additive group $\mathbb{Q}$ admit an epimorphism to a nontrivial finite group? Justify your answer.

Problem 4.

List all ideals of $\mathbb{F}_p[x]/(x^2 + x - 6)$ when

(a) $p = 7$

(b) $p = 5$.

Problem 5.

Let ρ be a representation of a finite group G on a vector space V and let $v \in V$.

(a) Show that averaging $\rho_g(v)$ over G gives a vector $\overline{v} \in V$ which is fixed by G.

(b) What can you say about this vector when ρ is an irreducible representation?

Problem 6.

If R is a commutative ring with identity, and S a multiplicative subset, then every ideal J of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R. Is I uniquely determined by J? Why or why not?

Spring 2013: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Let A be a *Boolean ring*, i.e., $a^2 = a$ for all $a \in A$. Show that the ring A is commutative.

Problem 2: Let R be a commutative ring with identity $1_R \neq 0$. Let $I \subset R$ be an ideal so that R/I is a division ring. Show that I is a maximal ideal in R.

Problem 3: Suppose that n and m are natural numbers. Show that the free group of rank n is isomorphic to the free group of rank m if and only if $n = m$.

Problem 4: Let E/K be a field extension of degree $[E : K] = 2^k$ for some $k \geq 1$. Let $f \in K[X]$ be a monic polynomial of degree 3 that has a root in E. Must f have a root in K?

Problem 5: Consider the multiplicative group $\mathbb{F}_{13}^{\times}$ of the field $\mathbb{F}_{13}$. Which elements generate the group, and which elements are squares in $\mathbb{F}_{13}^{\times}$?

Problem 6: Let G be a group. Prove or disprove the following statements.
(a) If G is abelian, then every finite-dimensional irreducible complex representation of G is one-dimensional.
(b) If G is finite and every irreducible complex representation of G is one-dimensional, then G is abelian.

Fall 2013: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Let $G \subset M_n(\mathbb{C})$ be a group of complex $n \times n$ matrices. Let V be the linear span of G, and let $V^\times$ be the set of invertible elements of V. Show that $V^\times$ is also a group.

Problem 2: Consider an attempt to make an $\mathbb{R}$-linear map

$$f : \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \qquad \text{or} \qquad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C},$$

in either direction given by the formula

$$f(x \otimes y) = x \otimes y.$$

In which direction is this map well-defined? Is it then surjective? Is it injective?

Problem 3: The dihedral group D_4 acts as the symmetries of a square in the plane $\mathbb{R}^2$ with coordinates x and y. Suppose that the corners of this square are at $(\pm 1, \pm 1)$. Then D_4 acts linearly, and it therefore has an induced action on the vector space V_n of homogeneous polynomials in x and y of degree n. Find the character of V_n viewed as a representation of D_4. (Note: the character will depend on n.)

Problem 4: Let G be a group with an odd number of elements that has a normal subgroup N with 17 elements. Show that N lies in the center of G.

Problem 5: Is it possible to have a field extension $F \subseteq K$ with $[K : F] = 2$, where both fields F and K are isomorphic to the field $\mathbb{Q}(x)$?

Problem 6: Compute $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.

Spring 2012: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Let A be a real $n \times n$ upper triangular matrix so that A commutes with its transpose A^T. Show that A is diagonal.

Problem 2. Suppose that G is a group which contains no index 2 subgroups. Show that every index 3 subgroup in G is normal.

Problem 3. Let F be a field and $F^{\times}$ be the multiplicative group of nonzero elements of F. Show that every finite subgroup of $F^{\times}$ is cyclic.

Problem 4.
Prove that $\mathbb{R}[X]/(X^2-1)\mathbb{R}[X] \approx \mathbb{R} \oplus \mathbb{R}$, but $\mathbb{R}[X]/(X^2-1)^2\mathbb{R}[X] \not\approx \mathbb{R} \oplus \mathbb{R}$.

Problem 5. Show that 9 and $6 + 3\sqrt{-5}$ do not have a greatest common divisor in $\mathbb{Z}[\sqrt{-5}]$.

Problem 6. Let F be a field, X an indeterminate, and let $F[[X]]$ denote the ring of formal power series with coefficients in F, where multiplication is defined as it is for polynomials. Prove that an element $s = a_0 + a_1X + \cdots \in F[[X]]$ is a unit in $F[[X]]$ if and only if $a_0 \neq 0$. Show that every ideal of $F[[X]]$ is of the form $X^nF[[X]]$ for some $n \geq 0$.

Fall 2012: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1.
a) Can a vector space V over an infinite field be a finite union

$$V = \bigcup_{i=1}^{k} V_i,$$

where for each i, $V_i \neq V$?
b) Can the group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ be a union of finitely many proper subgroups?

Problem 2. Let G be an abelian group with n generators. Show that every subgroup $H \subset G$ has a generating set consisting of at most n elements.

Problem 3. Let F be a field and let $P \subset F$ be the intersection of all subfields in F. Show that:
a) If F has characteristic 0 then $P \cong \mathbb{Q}$,
b) If F has characteristic $p > 1$ then $P \cong \mathbb{F}_p$.

Problem 4. Let R be a commutative ring and I an ideal in R. Prove or disprove: The set $\sqrt{I} = \{ a \in R : \exists n \in \mathbb{N},\, n > 0,\, a^n \in I \}$ is an ideal.

Problem 5. Find the number of field homomorphisms $\phi \colon \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{C}$.

Problem 6.
Consider the dihedral group $D_4 = \langle r, s : s^2 = r^4 = 1,\ rs = sr^{-1} \rangle$ of order 8.
a) Find the conjugacy classes of D_4.
b) Find the character table of D_4.

Spring 2011: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let N be an $m \times m$ square matrix of complex numbers. Prove that the following conditions are equivalent:
(a) $NN^* = N^*N$, i.e., N is normal.
(b) N can be expressed as $N = A + iB$, where A and B are self-adjoint matrices of order $m \times m$ satisfying $AB = BA$ (and $i = \sqrt{-1}$).
(c) N can be expressed as $N = R\Theta$, where R and Θ are matrices of order $m \times m$ satisfying $R\Theta = \Theta R$, Θ is unitary and R is self-adjoint.

Problem 2:
Prove that a finite group G is abelian if and only if all its irreducible representations are 1-dimensional.

Problem 3:
Let

$$SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

and let $PSL(2,\mathbb{Z})$ be the quotient group

$$PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\{\pm I\},$$

where I is the 2×2 identity matrix. Prove that $PSL(2,\mathbb{Z})$ is generated by the cosets of the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Hint: Note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-k} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - kc & b - kd \\ c & d \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

Problem 4:
Consider the group $G = \mathbb{Q}/\mathbb{Z}$. Show that for every natural number n the group G contains exactly one cyclic subgroup of the order n.

Problem 5:
Let R be a finite ring. Show that there exist n, m with $n > m$, so that

$$x^n = x^m$$

for all $x \in R$.

Problem 6:
Let J denote the ideal in $\mathbb{Z}[x]$ generated by 5 and the polynomial $p(x) = x^3 + x^2 + 1$. Determine if J is a maximal ideal.

Fall 2011: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
Show that there is no commutative ring with the identity whose additive group is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Problem 2:
Let $p \neq 2$ be prime and F_p be the field of p elements.
(a) How many elements of F_p have square roots in F_p?
(b) How many have cube roots in F_p?

Problem 3:
Prove that every finite group is isomorphic to a certain group of permutations (a subgroup of S_n for some n).

Problem 4:
Let G be the subgroup of S_{12} generated by $a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$ and $b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)$. Find the order of G, the number of conjugacy classes of G, and the character table of G.

Problem 5:
Prove or disprove: If the group G of order 55 acts on a set X of 39 elements then there is a fixed point.

Problem 6:
Prove or disprove: $(\mathbb{Z}/35\mathbb{Z})^* \cong (\mathbb{Z}/39\mathbb{Z})^* \cong (\mathbb{Z}/45\mathbb{Z})^* \cong (\mathbb{Z}/70\mathbb{Z})^* \cong (\mathbb{Z}/78\mathbb{Z})^* \cong (\mathbb{Z}/90\mathbb{Z})^*$. Here $(\mathbb{Z}/n\mathbb{Z})^*$ is the group of units in $\mathbb{Z}/n\mathbb{Z}$.

Spring 2010: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
1. Let R be a commutative ring with identity. Recall that an ideal I of R is said to be *radical* if for every $x \in R$ such that $x^n \in I$ for some n, we have $x \in I$. Prove that I is radical if and only if I is equal to the intersection of the prime ideals containing it. Hint for one direction: if I is radical and $x \notin I$ (equivalently, no power of x is in I), by Zorn's lemma there is a largest ideal J such that no power of x is in J (this means that no ideal with the same property strictly contains J, *not* that J contains every ideal with this property). Show that J is a prime ideal.

Problem 2:
Recall the definition of a projective module M over a ring R: Whenever A and B are two other R-modules, and whenever $f : M \longrightarrow A$ and $g : B \longrightarrow A$ where g is surjective, are module homomorphisms, then f factors as $f = g \circ h$. For instance, $M = \mathbb{R}$ is a module over the polynomial ring $\mathbb{R}[x]$, where x acts by multiplication by 0. Is this a projective module?

Problem 3:
Prove that the group $\langle x, y : x^2 = y^3 \rangle$ is not trivial.

Problem 4: Prove that every finite group of order greater than 2 has a non-trivial automorphism.

Problem 5: Prove that if R is an integral domain with a finite group of units $R^\times$, then the group of units is cyclic.

Problem 6: Give an example of an irreducible polynomial of degree n (for some n) over $\mathbb{Q}$ whose Galois group does not have $n!$ elements.

PRELIMINARY EXAMINATION FALL 2010

ALGEBRA

Problem 1. Let G be a group which admits a finite set of generators. Show that G is countable.

Problem 2. Let G be a finite group. Show that G embeds in $GL(n, \mathbb{Z})$ for some n.

Problem 3. Show that the (multiplicative) group of $n \times n$ upper-triangular matrices (with real entries), having diagonal elements that are non-zero, is solvable.

Problem 4. Consider the ring $R = \mathbb{Z}[x]$. Give an example, with a proof, of an ideal in R which is not principal and of an ideal that is not prime.

Problem 5. Let R be a ring with identity. Recall that $x \in R$ is called *nilpotent* if $x^n = 0$ for some n. Prove that if x is nilpotent, then $1 + x$ is invertible.

Problem 6. Let F be a nontrivial finite extension field of $\mathbb{R}$. Prove that F is isomorphic to $\mathbb{C}$. You may use the fundamental theorem of algebra.

Winter 2009: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
(a) Find a complex matrix M with

$$M^2 = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

(b) Note that $-M$ is another solution. How many such matrices are there?

Problem 2:
Show that there are at least two nonisomorphic, nonAbelian groups of each of the orders 24 and 30.

Problem 3:
Show that if a finite group has exactly three conjugacy classes (noting that the identity forms one of the three) then the group has at most six elements. (Hint: Consider showing that in a finite group the number of elements in any conjugacy class divides the number of elements in the group.)

Problem 4: Let P be some set of prime numbers in the usual integers. Find a commutative ring R containing the integers such that the primes (irreducible elements) in R are precisely the elements of P up to multiplication by units. (Hint: What are all primes in the ring $\mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^r} | a \in \mathbb{Z}, 0 \leq r \in \mathbb{Z}\}$?)

Problem 5: Let A be the group of rational numbers under addition, and let M be the group of positive rational numbers under multiplication. Determine all homomorphisms from A to M.

Problem 6: An element of the ring of n-adic integers $\mathbb{Z}_n$ is an infinite (to the left) string of base n digits written $\ldots a_3 a_2 a_1 a_0.$ with $0 \leq a_i \leq n-1$. Addition and multiplication are defined as usual for the integers written in base n, except that one must carry indefinitely. (Note for example that in $\mathbb{Z}_{10}$ the element $\ldots 99999.$ is -1.)

Prove that $\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_5$.

Fall 2009: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Recall that an integral domain R is said to be a *unique factorization domain* if every element $x \in R$ can be written as a product of irreducible elements $\prod_{i=1}^{m} p_i$, and if the p_i are uniquely determined up to reordering and multiplication by units. Show that if R is a unique factorization domain, then every irreducible element generates a prime ideal.

Problem 2. The field extensions $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{\sqrt{2}})/\mathbb{Q}(\sqrt{2})$ are both Galois (you do not need to prove this). Show that $\mathbb{Q}(\sqrt{\sqrt{2}})/\mathbb{Q}$ is not Galois. For concreteness, assume the square roots are positive.

Problem 3. Let A and B be linear transformations on a finite dimensional vector space V. Prove that the dimension of kernel(AB) is less than or equal to the dimension of kernel(A) plus the dimension of kernel(B).

Problem 4. Let G be a group and H and K subgroups such that H has finite index in G. Prove that the intersection of K and H has finite index in K.

Problem 5. Prove that the algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the algebra $\mathbb{C} \oplus \mathbb{C}$.

Problem 6. If V is a finite-dimensional linear representation of a group G, then by definition the character function $\chi(g)$ is the trace of the action of g. This is usually studied when V is a complex vector space, but it is well-defined over any field. Find an example of a non-trivial representation V of a group G over some field F, such that $\chi(g) = 0$ for all g. (Non-trivial means that not all of the elements of G act by the identity.)

Winter 2008: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Suppose that G is a finitely-generated group and $n \in \mathbb{N}$. Show that G contains only finitely many subgroups of index $\leq n$.

Problem 2. Let A be an $n \times n$ complex matrix. Prove of disprove:

a. A is similar to its transpose.

b. If the sum of the elements of each column of A is 1, then 1 is an eigenvalue of A.

Problem 3. Recall that if R is a ring, an R-module M is projective means: If $f : A \to B$ is a homomorphism between two other R-modules, and if $g : M \to B$ is a homomorphism, then there is always a solution $h : M \to A$ to the equation $g = fh$. Prove that among $\mathbb{Z}$-modules, the only cyclic module $\mathbb{Z}/n$ which is projective is $\mathbb{Z}/0 = \mathbb{Z}$.

Problem 4. 2. Prove that $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices, has no non-trivial two-sided ideals.

Problem 5. Let A and B be two abelian groups with 25 elements. There is more than one possibility for A up to isomorphism, and likewise for B. Since all abelian groups are $\mathbb{Z}$-modules, we may tensor A and B as $\mathbb{Z}$-modules. What are the possibilities for the number of elements of $A \otimes B$?

Problem 6. Prove or disprove: The field $\mathbb{C}(x)$ of rational functions with complex coefficients, is a transcendental (i.e., non-algebraic) extension of the field $\mathbb{C}$.

Winter 2008: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = (-1)^n x^n (1-x).$$

(a) Show that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[0,1]$.

(b) Show that $\sum_{n=0}^{\infty} |f_n|$ converges pointwise on $[0,1]$ but not uniformly.

Problem 2: Consider $X = \mathbb{R}^2$ equipped with the Euclidean metric,

$$e(x,y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{1/2},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} e(x,y) & \text{if } x,\ y \text{ lie on the same ray through the origin,} \\ e(x,0) + e(0,y) & \text{otherwise.} \end{cases}$$

Here, we say that x, y lie on the same ray through the origin if $x = \lambda y$ for some positive real number $\lambda > 0$.

(a) Prove that (X, d) is a metric space.

(b) Give an example of a set that is open in (X, d) but not open in (X, e).

Problem 3: Suppose that $\mathcal{M}$ is a (nonzero) closed linear subspace of a Hilbert space $\mathcal{H}$ and $\phi : \mathcal{M} \to \mathbb{C}$ is a bounded linear functional on $\mathcal{M}$. Prove that there is a unique extension of ϕ to a bounded linear function on $\mathcal{H}$ with the same norm.

Problem 4: Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a (complex) Hilbert space $\mathcal{H}$ with spectrum $\sigma(A) \subset \mathbb{C}$ and resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$. For $\mu \in \rho(A)$, let

$$R(\mu, A) = (\mu I - A)^{-1}$$

denote the resolvent operator of A.

(a) If $\mu \in \rho(A)$ and

$$|\nu - \mu| < \frac{1}{\|R(\mu, A)\|},$$

prove that $\nu \in \rho(A)$ and

$$R(\nu, A) = [I - (\mu - \nu)R(\mu, A)]^{-1} R(\mu, A).$$

(b) If $\mu \in \rho(A)$, prove that

$$\|R(\mu, A)\| \geq \frac{1}{d(\mu, \sigma(A))}$$

where

$$d(\mu, \sigma(A)) = \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is the distance of μ from the spectrum of A.

Problem 5: Let $1 \leq p < \infty$ and let $I = (-1, 1)$ denote the open interval in $\mathbb{R}$. Find the values of α as a function of p for which the function $|x|^\alpha \in W^{1,p}(I)$.

Problem 6: Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ denote the unit ball in $\mathbb{R}^3$. Suppose that the sequences $\{f_k\}$ in $W^{1,4}(\Omega)$ and that $\{\vec{g}_k\}$ in $W^{1,4}(\Omega; \mathbb{R}^3)$. Suppose also that there exist functions $f \in W^{1,4}(\Omega)$ and $\vec{g}$ in $W^{1,4}(\Omega; \mathbb{R}^3)$, such that we have the weak convergence

$$f_k \rightharpoonup f \text{ in } W^{1,4}(\Omega),$$
$$\vec{g}_k \rightharpoonup \vec{g} \text{ in } W^{1,4}(\Omega; \mathbb{R}^3).$$

Show that there are subsequences $\{f_{k_j}\}$ and $\{\vec{g}_{k_j}\}$ such that we have the weak convergence

$$\vec{D} f_{k_j} \cdot \operatorname{curl} \vec{g}_{k_j} \rightharpoonup \vec{D} f \cdot \operatorname{curl} \vec{g} \quad \text{in} \quad H^{-1}(\Omega)\,.$$

Notation for Problem 6. Here f is a scalar function and $\vec{g} = (g_1, g_2, g_3)$ are three-dimensional vector-valued function. $\vec{D}$ denotes the three-dimensional gradient $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\operatorname{curl} \vec{g} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \times \vec{g}$

As customary, we use $H^{-1}(\Omega)$ to denote the dual space of the Hilbert space $H_0^1(\Omega)$ consisting of those functions in $H^1(\Omega)$ which vanish on the boundary (in the sense of trace). Two useful identities are that

$$\begin{aligned} \operatorname{curl}(\vec{D} f) = 0 &\quad \text{for any scalar function } f\,, \\ \operatorname{div}(\operatorname{curl} \vec{w}) = 0 &\quad \text{for any vector function } \vec{w}\,, \end{aligned}$$

where $\operatorname{div} \vec{F} = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \partial_{x_3} F_3$ denotes the usual divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$.

Hint for Problem 6. Test $\vec{D} f_{k_j} \cdot \operatorname{curl} \vec{g}_{k_j}$ with a function $\psi \in H_0^1(\Omega)$ and use integration by parts to argue the weak convergence.

Fall 2008: September 23

Preliminary Examination in Algebra for the PhilosophiæDoctor degree from the University of California at Davis

Instructions:

1. *Each problem is worth 10 points.*
2. *Explain your answers clearly to receive credit.*
3. *Use a seperate sheet for each problem.*

Problems:

1. (a) Show that if $f(x) \in \mathbf{Q}[x]$ is an irreducible (nonconstant) polynomial then $\mathbf{Q}[x,y]/(f(x))$ is a principal ideal domain.
   (b) Find a generator for the ideal (x, y).
   (c) Show that $x^2 - y^3 \in \mathbf{Q}[x,y]$ is irreducible and $(x,y) \subseteq \mathbf{Q}[x,y]/(f(x))$ is not principal.

2. Assume that p is prime, D and P are subgroups of a finite group F with D normal and having index ($[F : D]$) relatively prime to p and P a p-group. Show that $P \subseteq D$.

3. Let M be a 3 by 3 matrix of complex numbers with characteristic polynomial $x^3 + 5x^2 + 3x + (9 - i)$.
   (a) Find the determinant of M^2.
   (b) Find the trace of M^2.
   (c) Find the characteristic polynomial of M^2.

4. Assume that R is an integral domain (a commutative ring with no zero divisors) and J is a nonzero ideal of R viewed as an R-module. Is J always, sometimes, or never a direct sum of two nontrivial R-submodules?

5. If H is a subgroup of a group G, then a subgroup $K \subseteq G$ is called a *complement* of H if K has exactly one element in every left coset of H.
   (a) Show that if H is normal, then all complements of H are isomorphic to each other.
   (b) Show that the inclusion of symmetric groups $S_3 \subset S_4$ has two complements which are not isomorphic.

6. Show that every sequence of finite abelian groups $\ldots, A_2, A_1, A_0$ is the homology of some chain complex

$$\cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

   of free abelian groups (that is if $d_i : C_i \to C_{i-1}$ are the maps above then $d_{i+1}d_i = 0$ and A_i is isomorphic to $\ker(d_i)/\mathrm{im}(d_{i+1})$).

Fall 2008: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Prove that the dual space of c_o is ℓ^1, where

$$c_o = \{x = (x_n) \text{such that} \lim x_n = 0\}.$$

Problem 2: Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Problem 3: Let $f \in L^1(R)$ and V_f be the closed subspace generated by the translates of f: $\{f(\cdot - y) | \forall y \in R\}$. Suppose $\hat{f}(\xi_0) = 0$ for some ξ_0. Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(R)$, then $\hat{f}$ never vanishes.

Problem 4: (a) State the Stone-Weierstrass theorem for a compact Hausdorff space X.

(b) Prove that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 5: For $r > 0$, define the dilation $d_r f : \mathbb{R} \to \mathbb{R}$ of a function $f : \mathbb{R} \to \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathcal{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, d_{1/r} \phi \rangle \qquad \text{for all test functions } \phi \in \mathcal{D}(\mathbb{R}).$$

(a) Show that the dilation of a regular distribution T_f, given by

$$\langle T_f, \phi \rangle = \int f(x) \phi(x) \, dx,$$

agrees with the dilation of the corresponding function f.

(b) A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ-distribution is homogeneous of degree -1.

(c) If T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n-1$.

Problem 6: Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \dots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Define $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} x_n^2 - x_n^4 \right\}$$

(a) Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \to \mathbb{R}$ equal to zero.

(b) Show that the second derivative of F at $x = 0$,

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R},$$

is positive-definite, meaning that

$$F''(0)(h, h) > 0$$

for every nonzero $h \in \ell^2(\mathbb{N})$.

(c) Show that F does not attain a local minimum at $x = 0$.

Winter 2007: PhD Algebra Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let R be a commutative ring with identity, and let I be an ideal of R. Under what conditions on I is R/I a field? An integral domain? A commutative ring with identity?

Problem 2. Let V be a vector space, and let A and B be a pair of commuting operators on V. Show that if W is an invariant subspace for A, then so are the spaces BW and $B^{-1}W := \{ v \in V \ : \ Bv \in W \}$.

Problem 3. Suppose the group G has character table

$$\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
3 & -1 & 0 & \zeta_5^3 + \zeta_5^2 + 1 & \zeta_5^4 + \zeta_5 + 1 \\
3 & -1 & 0 & \zeta_5^4 + \zeta_5 + 1 & \zeta_5^3 + \zeta_5^2 + 1 \\
4 & 0 & 1 & -1 & -1 \\
5 & 1 & -1 & 0 & 0,
\end{array}$$

where ζ_5 is a primitive 5-th root of unity (so $\zeta_5^4 + \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 = 0$).

(a) Prove that G is a simple group of order 60, and determine the sizes of its conjugacy classes.

(b) How does the tensor product of the two 3-dimensional irreps decompose into irreducibles?

Problem 4. Suppose that the group G is generated by elements x and y that satisfy $x^5y^3 = x^8y^5 = 1$. Is G the trivial group?

Problem 5. Let R be a principal ideal domain and $I \subset R$ an ideal. Prove that every ideal in the quotient ring R/I is a principal ideal. Show that R/I is not necessarily a principal ideal domain.

Problem 6.

(a) Give an example of a 4×4 complex matrix having only one eigenvalue, equal to 3, with the space of eigenvectors having dimension 2.

(b) Let us consider the set K of all matrices obeying the conditions of (a). The group $GL_4(\mathbb{C})$ acts on K by means of the transformations $\phi_A(X) = AXA^{-1}$. How many orbits does this action have?

Winter 2007: PhD Analysis Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0,1]) \to C([0,1])$ be a bounded linear operator. Suppose that $\|S(p)\| \leq 2$ for all polynomials p. Show that S is the zero operator.

Problem 2. For $p \geq 1$, let $l^p(\mathbb{N})$ be the set of sequences (x_n) such that

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

(a) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \subseteq l^q(\mathbb{N})$.

(b) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \neq l^q(\mathbb{N})$.

Problem 3. Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{x \to 0} \lim_{y \to 0} f(x,y) = \lim_{y \to 0} \lim_{x \to 0} f(x,y);$$

in particular, both limits exist. Does it follow that

$$\lim_{(x,y) \to (0,0)} f(x,y)$$

exists?

Problem 4. Let X be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < Cd(x,y)$ for all $x \neq y$. The function f is said to be a *weak contraction* if $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$, without the constant C. The contraction mapping theorem says that if f is a contraction, then it has a fixed point. Show that the theorem also holds when f is a weak contraction and X is compact.

Problem 5. Construct the Green's function for the Dirichlet boundary-value problem

$$-u'' + 4u = f, \quad u(0) = u(2) = 0.$$

Problem 6. Let U be a unitary operator on a Hilbert space. Prove that the spectrum of U lies on the unit circle.

Fall 2007: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Suppose that ρ is a complex matrix representation of a finite group G. Show that every matrix $\rho(g)$ is diagonalizable.

Problem 2. Consider the ring $\mathbb{R}[[x]]$ of formal power series in x with real coefficients. Namely, this is the set of all infinite series $a_0 + a_1 x + a_2 x^2 + ...$ with no conditions on convergence. What are the units (invertible elements) in this ring? What are the ideals?

Problem 3. If H is a subgroup of a group G, then the normalizer $N(H)$ of H is defined as the set of g in G such that $gHg^{-1} = H$. It is the largest subgroup of G that contains H as a normal subgroup. Let G be the symmetric group S_7; let $H \subset G$ be the cyclic subgroup generated by a 7-cycle.

Find the number of elements of the normalizer $N(H)$ of H in G.

Problem 4. Compute the number of groups of order ≤ 1029 each of which contains exactly three elements of order 3.

Problem 5. Show that the group $\mathbb{Q}$ of rational numbers (with respect to the addition operation) is not finitely generated.

Problem 6. Show that

$$\det(\exp(A)) = e^{tr(A)}$$

for every complex $n \times n$ matrix A, where $\exp(A)$ is defined as

$$\exp(A) = 1 + A + \frac{A^2}{2} + ... + \frac{A^k}{k!} + ...$$

PhD Algebra Preliminary Exam for 2005-06

Instructions: *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let field E be a finite extension of a field F, and let R be a subring of E that contains F. Prove that R is a field.

Problem 2. Let R be a commutative ring with a unit. Prove that the following two properties of R are equivalent:

(a) If $a, b \in R$ and $a + b$ is invertible, then either a or b is invertible.

(b) R is local, that is, R has a unique maximal ideal.

Problem 3. Describe all possible Jordan forms of an $n \times n$ matrix X obeying $X^n = 0$.

Problem 4. Show that $\mathbb{Q}$ (the additive group of rational numbers) is not finitely generated.

Problem 5. Determine all finitely generated abelian groups which have finite group of automorphisms.

Problem 6. Suppose that $H \subset G$ is a subgroup which is contained in every nontrivial subgroup of G. Show that H is contained in the center of G.

Analysis Preliminary Exam for 2005-06

Instructions: *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. (a) Prove that there is no continuous map from the closed interval $[0, 1]$ onto the open interval $(0, 1)$.

(b) Construct a continuous map from the interval $(0, 1)$ onto the interval $[0, 1]$.

Problem 2. Define the Fibonacci sequence (x_n) of integers by $x_1 = 1$, $x_2 = 1$ and

$$x_{n+1} = x_n + x_{n-1}, \qquad n = 2, 3, \ldots.$$

Let $r_n = x_{n+1}/x_n$ be the ratio of successive terms. Prove that r_n converges to ϕ as $n \to \infty$, where ϕ is the "golden ratio"

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

Problem 3. Suppose that X is a complete metric space with metric d. Let $(F_n)_{n=1}^{\infty}$ be a decreasing (i.e. $F_{n+1} \subset F_n$ for all n) sequence of nonempty, closed subsets of X such that $\operatorname{diam} F_n \to 0$ as $n \to \infty$. Here,

$$\operatorname{diam} F = \sup\{d(x, y) \mid x, y \in F\}$$

denotes the diameter of F. Prove that the intersection $\cap_{n=1}^{\infty} F_n$ consists of a single point.

Problem 4. Let $f, g \in L^2(\mathbb{T})$, where $\mathbb{T}$ is the circle, identified with the quotient of $\mathbb{R}$ by the subgroup $2\pi\mathbb{Z}$. Let $*$ denote the convolution on $L^2(\mathbb{T})$. Show that the identity

$$f * g = \frac{1}{2}(f * f + g * g)$$

holds if and only if $f = g$.

Problem 5. Let $\{u_k \mid k \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. Find (i.e. characterize) all sequences of scalars $\{a_k \mid k \in \mathbb{N}\}$ such that the set $\{a_k u_k \mid k \in \mathbb{N}\}$ is compact in $\mathcal{H}$.

Problem 6. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a compact linear operator on a complex Hilbert space $\mathcal{H}$. If $\lambda \in \mathbb{C}$ is nonzero, prove that the range of $\lambda I - T$ is closed.

Fall 2006: PhD Algebra Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let G be a matrix group, and let $g \in G$ be an element with $\det(g) \neq 1$. Show that $g \notin G'$, the commutator group of G.

Problem 2. Let $A : V \to V$ be an operator on a finite-dimensional vector space V. Suppose A has characteristic polynomial $x^2(x-1)^4$ and minimal polynomial $x(x-1)^2$. What is the dimension of V? What are the possible Jordan forms of A?

Problem 3. Show that $\mathbb{Z}$ is a principal ideal domain.

Problem 4. Let G denote a finite abelian group. Let us consider the set G^* of all homomorphisms of the group G into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers.

(a) Check that G^* can be considered as a group with respect to the operation of multiplication of homomorphisms.

(b) Prove that the group G^* is isomorphic to the group G.

Problem 5. Let us assign to every nonsingular complex 2×2 matrix A a transformation ϕ_A of the vector space Mat_2 of complex 2×2 matrices defined by the formula

$$\phi_A(X) = AXA^{-1}.$$

(a) Check that this formula specifies an action of the group $GL_2(\mathbb{C})$ of nonsingular complex matrices on Mat_2; moreover, it specifies a linear representation of this group.

(b) Prove that this representation is reducible.

(c) For every orbit of the above action, write down one element in that orbit, and find the corresponding stabilizer.

Problem 6. Consider the dihedral group D_9 (the group of isometries of regular 9-gons).

(a) Prove that D_9 cannot be represented as a direct product of two non-trivial groups.

(b) Determine if D_9 is solvable.

Fall 2006: PhD Analysis Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0,1]\times[0,1] \to \mathbb{R}$ be a given continuous function. Let $T_k : C([0,1]) \to C([0,1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x,y)f(y)\,dy$.

(a) Show that T_k is a bounded operator.

(b) Find an expression for $\|T_k\|$ in terms of k.

(c) What is $\|T_k\|$ if $k(x,y) = x^2y^3$?

Problem 2. Let X be a metric space.

(a) Define X is sequentially compact.

(b) Define X is a complete metric space.

(c) Prove that a sequentially compact metric space X is complete.

(d) Let $B = \{\, x \;:\; \|x\|_2 \le 1 \,\}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that B is not sequentially compact.

Problem 3. Give an example of a Banach space X and a sequence (x_n) of elements in X such that $\sum_{n=1}^\infty x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^\infty |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n\in\mathbb{Z}}$ be the Fourier coefficient sequence of f; here, $\mathbb{T} := \{\, z \in \mathbb{C} \;:\; |z| = 1 \,\}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that f is continuous? (In other words, is there a continuous function that is equivalent to f in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions T of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^{\mathbf{1}})$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0,1])$ *symmetric*, *essentially self-adjoint*, *self-adjoint*? Justify your answers.

(a) $D_A = C^1[0,1]$
(the space of continuously differentiable complex-valued functions on $[0,1]$)

(b) $D_A = \{\, f \in C^1[0,1] \;:\; f(0) = f(1) \,\}$

(c) $D_A = \{\, f \in C^1[0,1] \;:\; f(0) = f(1) = 0 \,\}$

Algebra Prelim Exam for 2004-05

Instructions: *explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let V be a nonzero finite-dimensional complex vector space, and let $f, g\colon V \to V$ be two linear maps. Prove that there exists a non-zero vector $v \in V$ such that the vectors $f(v), g(v)$ are collinear (that is, $\dim(\mathrm{Span}(f(v), g(v))) \leq 1$).

Warning: neither of f, g is assumed to be non-singular.

Problem 2. Prove that an infinite simple (not having proper normal subgroups) group does not have proper subgroups of finite index.

Problem 3. Let G be a finitely generated abelian group. Prove that there are no non-zero homomorphisms $\mathbb{Q} \to G$ (here $\mathbb{Q}$ is the additive group of rational numbers).

Problem 4. Prove of disprove: $\mathbb{C}[x, y]$ is a PID (Principal Ideal Domain).

Problem 5. Give examples of each of the following

a) a finite nonabelian group

b) an infinite nonabelian group

c) a group that is not finitely generated

d) a group that is not solvable

Problem 6.

a) Construct infinitely many non-isomorphic quadratic extensions of $\mathbb{Q}$.

b) Use (a) to show that the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ does not have finitely generated abelianization.

Here $\mathbb{Q}$ is the field of rational numbers.

Analysis Prelim Exam for 2004-05

Instructions: *explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function such that there exists a limit

$$\lim_{x \to 0} \frac{f(x)}{x^2} = L \in \mathbb{R}.$$

Does it follow that the second derivative $f''(0)$ exist and equals L? Give a proof or a counter-example.

Problem 2. For functions from $[0, 1]$ to $\mathbb{R}$ do the following:

a) Define what it means for a sequence of functions to converge uniformly.

b) Explain what it means for a sequence of functions to be equicontinuous.

c) Does every equicontinuous sequence of functions converge uniformly to a continuous function? Is the converse true? Give examples or prove.

Problem 3. Define two sequences of functions, (f_n) and (g_n), on the interval $[0, 1]$ as follows:

$$\begin{aligned} f_n(x) &= (1 + \cos 2\pi x)^{1/n}, \quad n \geq 1 \\ g_n(x) &= (1 + \frac{1}{2} \cos 2\pi x)^{1/n}, \quad n \geq 1 \end{aligned}$$

a) What are the pointwise limits, f and g, of the sequences (f_n) and (g_n) respectively?

b) For each sequence, determine whether the convergence is uniform. Explain your answer.

Problem 4. Let X and Y be a topological spaces. Prove that if $f : X \to Y$ is continuous and X is compact, then $f(X)$ is also compact.

Problem 5. Let X be a normed linear space and let X^* be its topological dual. Suppose that for $x, y \in X$ are such that for all $\varphi \in X^*$, $\varphi(x) = \varphi(y)$. Prove that $x = y$.

Problem 6. Consider the following equation for an unknown function $f : [0, 1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x - y)^2 f(y)\, dy + \frac{1}{2} \sin(f(x)) \tag{1}$$

Prove that there exists a number $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all continuous functions g on $[0, 1]$, the equation (1) has a unique continuous solution.