Problem 1 Consider a rigid, flat object, of mass m and length ℓ, attached to a torsional spring of stiffness κ in the presence of wind of speed v (see Fig. 1). The equations of motion are

$$\frac{m\ell^2}{4}\frac{d^2\theta}{dt^2} = -\kappa\theta + vc\frac{\ell}{2}\sin(\theta)$$

where c is a drag constant so that vc has units of force. Assume that all constants listed above are positive (but keep in mind that $\theta(t)$ may be negative).

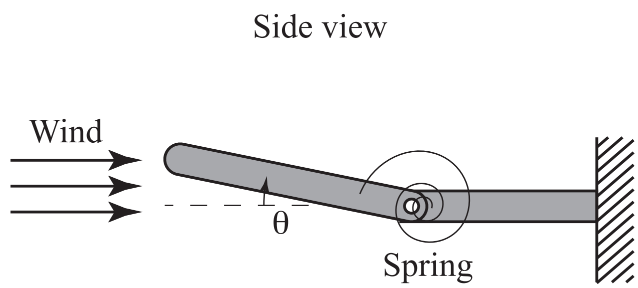


Figure 1:

(a) Use non-dimensionalization to show that the qualitative behavior of the system is defined by a single non-dimensional parameter.
(b) Show that there is a conserved quantity.
(c) As wind speed v increases from 0, find a critical value of the non-dimensional parameter at which a bifurcation occurs, and identify the type of bifurcation.
(d) Sketch the phase portrait at **i.** a wind speed just below the bifurcation and **ii.** a wind speed just after the bifurcation.

Problem 2 Consider the following predator-prey model:

$$\frac{dx}{dt} = x\big(x(1-x) - y\big) \qquad \frac{dy}{dt} = y(x-a)$$

where x is the (positive) non-dimensional population of prey, y is the (positive) non-dimensional population of predators, and a is a (positive) non-dimensional parameter.

(a) Sketch the null-clines in the first quadrant, $x, y \geq 0$.
(b) Find and classify all fixed points
(c) Find and classify all bifurcations that occur as a varies (assume $a > 0$).
(d) Show that a stable limit cycle exists for some values of a.

Problem 3 An *annular* plate with inner and outer radii $a < b$, respectively, is held at temperature B at its outer boundary and satisfies the boundary condition $\dfrac{\partial u}{\partial r} = A$ at its inner boundary, where A, B are constants. Find the temperature if it is at a *steady state*.

[Hint: It satisfies the two-dimensional Laplace equation and depends only on r. You can also use the fact that the Laplace operator can be expressed in the polar coordinate (r, θ) as: $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$.]

Problem 4 Let ω be positive, but *not* an integer multiple of π and consider the following boundary value problem on the unit interval $[0,1]$:

$$f'' + \omega^2 f = g, \quad f'(0) = 0 = f'(1).$$

(a) Find the *Green's function* for this boundary value problem.

(b) Discuss what happens if we try this with $\omega = 0$?

Problem 5 Let A be a symmetric matrix and let λ_0 be a simple (i.e. multiplicity one) eigenvalue of A with corresponding eigenvector $\mathbf{v}_0$. Derive an expression for the eigenvalue, λ, up to order ϵ in the limit of small ϵ to the problem

$$A\mathbf{v} + \epsilon\mathbf{F}(\mathbf{v}) = \lambda\mathbf{v}$$

that is λ_0 at leading order.

Problem 6 The van der Pol oscillator,

$$\begin{aligned}\epsilon\dot{u} &= v + u - \frac{u^3}{3},\\ \dot{v} &= -u,\end{aligned}$$

exhibits periodic relaxation oscillations. The oscillation exhibits two time scales (a fast and slow time scale) for small ϵ.

Let $f(u) = u^3/3 - u$. The following information about f may be helpful:

$$\begin{aligned}f'(\pm 1) &= 0\\ f(\pm 1) &= \mp 2/3\\ f(\pm 2) &= \pm 2/3\end{aligned}$$

(a) Draw the nullclines in the phase plane (uv-plane), sketch the the limit cycle for small ϵ, and label the regions of fast and slow dynamics on the limit cycle.

(b) Compute the period of the oscillation at leading order as $\epsilon \to 0$.

End of the exam.

Problem 1 Consider the oscillator equation

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0,$$

where $F(x, \dot{x}) < 0$ if $r \leq a$ and $F(x, \dot{x}) > 0$ if $r \geq b$ with $r^2 = x^2 + \dot{x}^2$ and $a < b$. Show that there is at least one closed orbit in the region $a < r < b$.

Problem 2 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x(x - 2y)$$
$$\frac{dy}{dt} = y(2x - y)$$

(a) Show that $(x, y) = (0, 0)$ is the unique fixed point of the system.

(b) Use linear stability analysis to classify the fixed point at $(x, y) = (0, 0)$. What can you conclude about the stability of $(0, 0)$ based on this analysis?

(c) Sketch the phase portrait of the system and describe the stability of $(x, y) = (0, 0)$.

Problem 3 Suppose you are given a string of length L. Suppose you arrange it to lie along a function, $f(x)$, where $f(0) = 0$. Of all possible potential arrangements of the string, which one maximizes the volume enclosed by it, V, when it is rotated about the x-axis?

DO NOT look for a closed form solution. Instead, leave your answer as a differential equation, boundary conditions, and a sufficient number of constraint equations to allow a clever person with a computer to find a solution.

Problem 4 A plucked string, fixed at both ends, obeys the differential equation

$$u_{tt} = c^2 u_{xx} - a u_t$$

with boundary conditions $u(0,t) = u(L,t) = 0$, and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$. In these equations, u is the local displacement of the string at position x, and c is a constant (t is time). When the constant a is zero, this is the wave equation; here, you will examine the effect of $a > 0$.

(a) Write the solution to the differential equation.

(b) What happens to the solution as $t \to \infty$?.

(c) Give a possible physical interpretation of the term $a u_t$.

Problem 5 Consider projectile motion with air resistance. The (dimensional) ODE for $x(t)$, the height of the object is

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} - \frac{k}{x+R}\frac{dx}{dt}, \tag{1}$$

where g is the gravitational constant, R is the radius of the earth, and k is a non-negative constant related to the air resistence. Suppose an object is launched from the surface ($x(0) = 0$) at a low velocity $\left.\frac{dx}{dt}\right|_{t=0} = v_0$ (with v_0 small).

1. Non-dimensionalize Equation (1) by finding appropriate re-scalings of x and t and define (two) small parameters in terms of your scaling choices. [HINT: Your choice of scaling should give the familiar physical problem valid when the initial velocity or displacement is much smaller than R and air resistance is negligible.]
2. Using the non-dimensionalized equations, find the leading order asymptotic expansion for the solution. [HINT: Your expansion should be in orders of the small parameter you defined above that is *independent* of the air-resistance parameter k.]
3. What equation would you need to solve to find the solution to the next highest order in the small parameter, include initial conditions, but DO NOT solve the equation.

Problem 6 Determine the first terms in the inner and outer expansions for the following boundary value problem:

$$\epsilon y'' - (2x+1)y' + 2y = 0$$

with $y(0) = 1, y(1) = 0$, and $\epsilon \ll 1$. Construct a first-order uniformly valid expansion for $y(x)$.

End of the exam.

Problem 1 Suppose a bead, of mass m, slides frictionlessly on a hoop of radius R. If we then spin the hoop at constant angular velocity ω about an axis parallel to the force of gravity (see Fig. 1), the bead obeys the following non-linear second order differential equation

$$\frac{d^2\theta}{dt^2} - \omega^2 \sin(\theta)\cos(\theta) + \frac{g}{R}\sin(\theta) = 0$$

where g is the acceleration of gravity, $\theta(t)$ is the bead's angular position on the hoop (with $\theta = 0$ being at the bottom), and t is time.

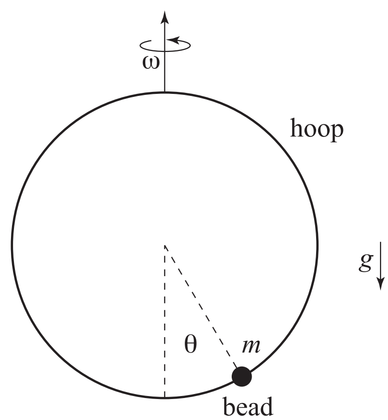


Figure 1:

a) Use non-dimensionalization to show that the qualitative behavior of the system is defined by a single non-dimensional parameter.

b) Find all fixed points, determine their stability and classify them as a function of that parameter.

c) Sketch a bifurcation plot (i.e., sketch the fixed points as a function of the parameter, indicate the stability of the fixed points, and label any bifurcations that occur). Use the Lyapunov definition of stability for this part.

It may or may not be useful to know that the energy of the system can be written as

$$E = mg\,(R - R\cos(\theta)) + \frac{m}{2}\left(R^2\sin^2(\theta) + R^2\left(\frac{d\theta}{dt}\right)^2\right)$$

The Lyapunov definition of stability is that a fixed point is stable if all trajectories starting sufficiently close to the fixed point remain within an arbitrarily small distance of the fixed point.

Problem 2 A solid box, with sides of unequal length, obeys Euler's equations when tossed in the air:

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{0}$$

where, for simplicity, we neglect gravity. In this equation, $\boldsymbol{I}$ is the inertia tensor (defined below) and $\boldsymbol{\omega}$ is the angular velocity vector.

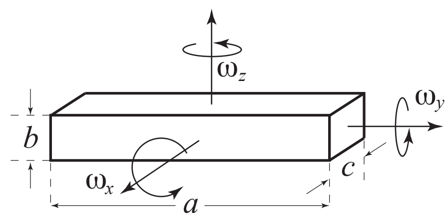


Figure 2:

For the box above (Fig. 2), with length a, height b, and width c, the inertia tensor (in Cartesian coordinates) is

$$\boldsymbol{I} = \begin{bmatrix} \frac{m}{12}(a^2+b^2) & 0 & 0 \\ 0 & \frac{m}{12}(c^2+b^2) & 0 \\ 0 & 0 & \frac{m}{12}(a^2+c^2) \end{bmatrix}$$

and the corresponding angular velocity vector is

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

For the following, assume $a > c > b$, and $||\boldsymbol{\omega}|| = 1$.

a) Find all fixed point(s).

b) Use linear stability analysis to classify the fixed point(s), i.e., stable node, unstable node, center, stable spiral, unstable spiral, saddle.

Problem 3 Find a planar curve $(x, y) = (x(t), y(t))$ that minimizes the following functional:

$$I = \int_0^1 m\left(\frac{\dot{x}^2 + \dot{y}^2}{2} - g y\right) \mathrm{d}t,$$

where m, g are positive constants, $(x(0), y(0)) = (0,0)$, and $(x(1), y(1)) = (a, 0)$.

[Physically, this is a problem to find a trajectory of a projectile of mass m that starts at $(0,0)$ and hits at $(a,0)$ at time $t = 1$ under gravity.]

Problem 4 Consider the *Regular Sturm-Liouville Problem* on the unit interval $[0,1]$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d} x^2} + \lambda f = 0, \quad f(0) = 0, f(1) + f'(1) = 0.$$

a) Find the eigenvalues and eigenfunctions of this RSL system.

[Hint: Those eigenvalues are the solutions of some transcendental (also known as *secular*) equation.]

b) Expand the constant function 1 on $[0,1]$ into the series of the eigenfunctions obtained in Part (a).

Problem 5 The modified Bessel function $I_n(x)$ for n an integer has the integral representation

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} \exp\left(x\cos\theta\right)\cos(n\theta)\,d\theta.$$

Find the leading order asymptotic expansion for $I_n(x)$ as $x \to \infty$. You may find the following integrals useful:

$$\int_{-\infty}^{\infty} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}},\ a > 0; \qquad \Gamma(x) = \int_0^{\infty} t^{x-1}\mathrm{e}^{-t}\,dt.$$

Problem 6

a) Show that the all of the solutions to

$$\ddot{u} + u + \epsilon u^3 = 0, \quad \epsilon \geq 0$$

are periodic in time.

[Hint: One could show that all nontrivial trajectories in the phase plane are closed curves.]

b) For $\epsilon = 0$, the period of the oscillation is 2π. Find the leading ϵ-dependent correction to the period in the limit of of small ϵ for solutions that pass through the point

$$u(t_0) = A, \quad \dot{u}(t_0)(0) = 0,$$

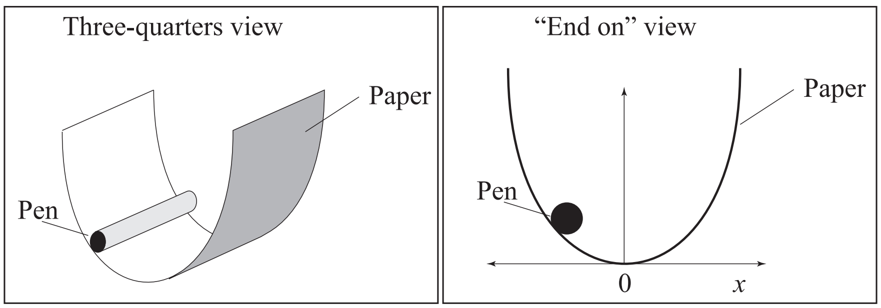
where $t = t_0$ is some time.

End of the exam.

Problem 1 Consider the following system:

Step 1: Take a sheet of paper and hold it in a "U" shape.
Step 2: Place a pen or pencil near the bottom of the "U", but slightly off to one side.
Step 3: Let go, and watch the pen or pencil move back and forth.

Here's a sketch of the set-up.



When you do this experiment, the horizontal position of the pen (x in the sketch on the right) is a function of time t, and obeys the following equation (assuming conservation of energy)

$$\ddot{x} = \frac{-f'(x)}{a(1+f'(x))} - \frac{f''(x)}{2(1+f'(x))}(\dot{x})^2 \tag{1}$$

where $f(x)$ is a function that gives the height of the paper (in m) as a function of x (you may assume it and all of its derivatives are continuous), and a is a constant with units of s^2/m. Note that the dot indicates a time derivative and prime indicates a derivative with respect to x, which is measured in m.

a) Find all fixed points and determine their stability (your answer should depend on a, $f(x)$ and/or its derivatives). Note, for this question, use the Lyapunov definition of stability, where a fixed point is stable if trajectories that start sufficiently close (but not exactly at) the fixed point remain within some small neighborhood of the fixed point. You may assume that there is no point at which $f'(x) = f''(x) = 0$.

b) You might expect that the pen will oscillate about a stable fixed point. Find the period of this oscillation (your answer should depend on a, $f(x)$ and/or its derivatives). Your answer should include units.

Problem 2 Suppose you are studying the interaction of two proteins. The concentration of the first protein is $p(t)$ and the concentration of the second protein is $w(t)$. They interact via the following equations:

$$\begin{aligned}\dot{p} &= A_p \frac{p^2}{K_p^2 + p^2} - kwp \\ \dot{w} &= A_w \frac{w}{K_w + w} - kwp\end{aligned}$$

In each equation, the first term models the formation of protein, and the second term models the breakdown of the protein. Concentration is measured in units of number per liter and time is measured in seconds. The constants K_p and K_w then have units of concentration; the constants A_p and A_w have units of concentration per second; and k has units of inverse concentration per second.

a) Suppose that you know $K_p/K_w = \varepsilon$ (where ε is a small number), $A_w/(K_p^2 k) = \alpha/\varepsilon$ (where α is of order 1) and $A_p/(K_p^2 k) = \beta$ (where β is of order 1). Non-dimensionalize the equations and write them in terms of the appropriate non-dimensional variables and the non-dimensional constants ε, α and β.

b) Simplify the equations by expanding in ε and neglecting all terms of order ε.

c) Identify all fixed points and determine their stability. Discuss any bifurcations that may occur.

d) Sketch a phase portrait of the system. On your plot, be sure to (1) identify and classify all fixed points, (2) draw all null clines, (3) indicate the qualitative flow direction, and (4) sketch a few sample trajectories.

Problem 3 Let L be the differential operator

$$Lu = \left(1 + x^2\right) u'' - 2xu' \qquad 0 < x < 1$$

with boundary conditions $Bu = 0$ given by $u'(0) = 0,\ u'(1) = 0$.

(a) Find the adjoint L^* of L in $L^2(0,1)$ and the adjoint boundary conditions $B^* u = 0$.

(b) What are the solutions of the homogeneous boundary value problem $Lu = 0$, $Bu = 0$? What is the dimension of the null space?

(c) What are the solutions of the homogeneous adjoint boundary value problem $L^* u = 0$, $B^* u = 0$? What is the dimension of the null space?

Problem 4 (a) Let $\Omega = \{(x,y) \in \mathbb{R}^2 : -a < x < a,\ 0 < y < b\}$ be a rectangle. Use separation of variables to solve the boundary value problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \qquad (x,y) \in \Omega \\ &u(x,0) = 0, \qquad u(x,b) = e^{-x^2}, \\ &u_x(-a,y) = 0, \qquad u_x(a,y) = 0. \end{aligned}$$

(b) What is a physical interpretation of this problem? What is the approximate limiting behavior of the solution as $b \to 0$?

Problem 5

The equation for displacement, $q(x)$, of a nonlinear beam, on an elastic foundation and with an additional small forcing, is

$$q'''' - \kappa q'' + c^2 q = \epsilon \sin(\pi x), \quad \text{for } 0 < x < 1,$$
$$\kappa = \frac{1}{4}\int_0^1 (q_x)^2 \, dx,$$
$$q(0) = q''(0) = q(1) = q''(1) = 0,$$

where c is a positive constant. Find a two-term expansion of the solution for small ϵ.

Problem 6 The dimensionless equation of motion of a frictionless pendulum is

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

In the limit of small amplitude (e.g. denote the amplitude of the θ as ϵ), the period is 2π to leading order. Compute the next term in the expansion of the period for small amplitude.

End of the exam.

Problem 1 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x(1-x^2-y^2) - 2y(1+x)$$
$$\frac{dy}{dt} = y(1-x^2-y^2) + 2x(1+x).$$

(a) Use the function $V(x,y) = (1-x^2-y^2)^2$ like a Lyapunov function to prove the existence of an *asymptotically stable* closed orbit.

(b) Is the asymptotically stable closed orbit a limit cycle? Briefly justify your answer.

Problem 2 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x^2 - y$$
$$\frac{dy}{dt} = 2\alpha x - y - \beta$$

with the parameters $\alpha, \beta > 0$.

(a) Find and classify *all* bifurcations of steady states that occur in the system. (That is, identify all saddle-node, pitchfork, transcritical, and/or Hopf bifurcations. For any pitchfork or Hopf bifurcations, you do NOT have to determine whether they are super- or sub-critical).

(b) Plot the stability diagram (i.e., two-parameter bifurcation diagram) for the system in the α,β-plane. A codimension-2 bifurcation called a Taken-Bogdonov bifurcation occurs at $\alpha = 1/2, \beta = 1/4$. *Very briefly* describe what happens at this point.

Problem 3 Suppose you are given a string of length L. Of all possible potential arrangements of the string, which one maximizes the area enclosed by it, A?

Assume (1) that the shape of the string is symmetric; and
(2) that half of the string (of length $L/2$) can be described by the function $f(x)$, defined for $a \le x \le b$, such that $\int_a^b f(x)dx = A/2$.

Problem 4 A uniform, isotropic, linear-elastic beam of length L, subject to small transverse displacements has action $\mathscr{L}$,

$$\mathscr{L} = \int_0^L \left(-a^2 \frac{1}{2} u_x^2 + \frac{1}{2} u_t^2\right) dx,$$

where u is the local displacement at position x along the beam, and a is a constant (t is time, and the equation is non-dimensionalized).

(a) Derive a partial differential equation for the function, $u(x,t)$, that minimizes the action $\mathscr{L}$.

(b) Suppose that the beam is fixed at one end ($u(0,t) = 0$), and free at the other ($u_x(L,t) = 0$). Solve the PDE you derived in part (a) for arbitrary initial displacement ($u(x,0) = f(x)$) and zero initial velocity ($u_t(x,0) = 0$).

(c) Find the solution for the case where the beam is struck at the free end ($u_t(x,0) = b \cdot \delta(x-L)$, where b is an arbitrary positive constant, and $\delta(x)$ is the Dirac delta function).

Note, it may be useful to recall the property $\int_a^b f(x)\delta(x-c)\,dx = f(c)$, if $a < c < b$.

Problem 5 Use the WKB method to find an approximate solution to the following problem

$$\begin{cases} \varepsilon y'' + 2y' + 2y = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

HINT: Assume $y(x) = g(x)f(x)$ for some function $g(x)$ (that you must determine) to put the equation into the standard WKB form, namely $f''(x) - q(x)f(x) = 0$.

Problem 6 Assuming $\lambda \gg 1$, derive an approximation to the integral

$$I(\lambda) = \int_{-1}^{2} (1 + x^2) e^{-\lambda x^6} \, dx.$$

HINT: You may write your approximation in terms of the Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx.$$

End of the exam.

Applied Mathematics Preliminary Exam
(Spring 2016)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the one-dimensional dynamical system

$$\frac{dx}{dt} = \mu x - 2x^2 + x^3$$

where $\mu \in \mathbb{R}$ is a parameter and $x(t) \in \mathbb{R}$.

(a) Determine the equilibria of the system and for what ranges of μ they exist.

(b) Determine the stability of the equilibria in (a).

(c) Sketch the bifurcation diagram for this system, using a solid line to denote a branch of stable equilibria and a dashed line to denote a branch of unstable equilibria. Classify the bifurcations that occur as μ increases from $-\infty$ to ∞.

Problem 2 (a) Show that the second order ODE

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + x = 0$$

can be put in the Hamiltonian form

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \tag{1}$$

by defining

$$p = e^{2x}\frac{dx}{dt}.$$

What is $H(x, p)$?

(b) Sketch the phase plane of the resulting Hamiltonian system (1).

Problem 3 Suppose a perfectly flexible rope of length $2a$ with uniform density ρ hangs under gravity from two fixed points $(-b,0)$ and $(b,0)$ in the xy-plane where $b < a$ and the gravity points downward (i.e., the negative y direction). Find the shape of this rope, $y = y(x)$, that minimizes the potential energy

$$V = \rho g \int_{-b}^{b} y\sqrt{1+(y')^2}\,\mathrm{d}x.$$

[Hint: The constraint is of course the arclength of the rope must be $2a$.]

Problem 4 Let $f(\theta)$ be the 2π-periodic function such that $f(\theta) = \mathrm{e}^{\theta}$ for $-\pi < \theta \le \pi$, and let $\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta}$ be its Fourier series; thus $\mathrm{e}^{\theta} = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta}$ for $|\theta| < \pi$.

(a) Compute c_n, $n \in \mathbb{Z}$, explicitly.

(b) If we formally differentiate this equation, we obtain $\mathrm{e}^{\theta} = \sum_{n=-\infty}^{\infty} \mathrm{i}nc_n \mathrm{e}^{\mathrm{i}n\theta}$. But then, $c_n = \mathrm{i}nc_n$ or $(1-\mathrm{i}n)c_n = 0$, so $c_n = 0$ for all n. This is obviously wrong; where is the mistake?

Problem 5 Find a one-term approximation, that is valid for long time scales, of the solution to the following differential equation

$$\varepsilon \frac{d^2x}{dt^2} + \varepsilon \frac{dx}{dt} + x = \cos(t)$$

for $t > 0$, with initial conditions $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$.

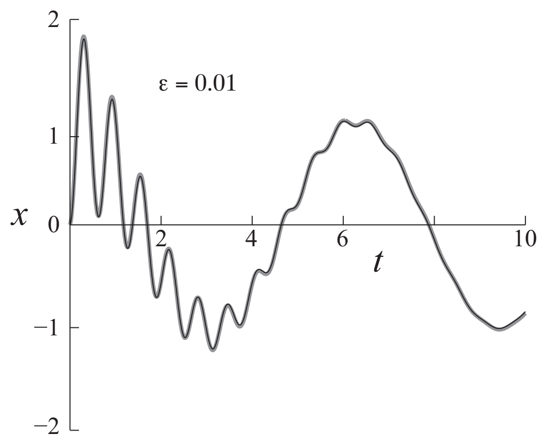


Figure 1: This is a numerical solution of the equation with $\varepsilon = 0.01$ (gray), plotted with my solution for the one term approximation, valid for long time scales (black).

Problem 6 Friedrichs' (1942) model problem for a boundary layer in a viscous fluid is

$$\varepsilon \frac{d^2y}{dx^2} = a - \frac{dy}{dx}$$

for $0 < x < 1$ and $y(0) = 0, y(1) = 1$, and a is a given positive constant.

After finding the first term of the inner and outer expansions, derive a composite expansion for the solution to this problem.

Problem 1 The SIR model is a simple and sometimes accurate way to describe the spread of a disease in a population. One variant of the model is given by the following three equations:

$$\begin{aligned}\frac{dS}{dt} &= a(I+R+S) - aS - bSI \\ \frac{dI}{dt} &= bSI - aI - cI \\ \frac{dR}{dt} &= cI - aR \end{aligned} \tag{1}$$

where S is the number of susceptible individuals, I the number of infected individuals and R the number of recovered individuals in the population and t is time.

The parameters are defined as follows:

a is the birth rate and also the death rate. Since these rates are equal, the population maintains a constant size, $R + I + S = N$, where N is a constant.

b is the transmission likelihood. When a susceptible and infected individual meet, the susceptible becomes infected with some probability. The parameter b defines the rate that susceptible and infected individuals meet and the infection is transmitted.

c is the recovery rate. An infected individual recovers at this rate, and then is immune to the disease.

a. Using a and N to define your time and population scales, respectively, non-dimensionalize the three differential equations.

Given the appropriate non-dimensionalization, and using the constraint that the population maintains a constant size, the equations become

$$\begin{aligned}\frac{dx}{dT} &= 1 - x - \alpha xy \\ \frac{dy}{dT} &= \alpha xy - (1+\beta)y \end{aligned} \tag{2}$$

where x is the probability that an individual is susceptible, y is the probability that an individual is infected, and the probability that an individual is resistant (z) can be determined from the constraint $x + y + z = 1$.

b. Find all fixed points (x^*, y^*) and determine their stability for all combinations of $\alpha, \beta > 0$.

c. Suppose that $\beta = 1$. A bifurcation occurs as α changes. Classify this bifurcation, and sketch a phase portrait before and after the bifurcation.

Problem 2 Consider the following mechanical system.

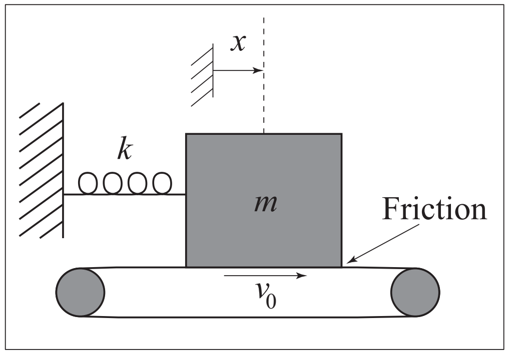


Figure 1: Mechanical system for problem 2.

A block, of mass m, sits on a conveyer belt moving at velocity v_0. The mass is attached to a wall with a linear spring of stiffness k. The position of the mass, x, as a function of time, t, obeys the following differential equation

$$m\frac{d^2x}{dt^2} = -kx - f(\dot{s})$$

where f is the frictional force that the conveyer belt applies to the block and $\dot{s}$ is the velocity of the block relative to the belt, $\dot{s} = \dfrac{dx}{dt} - v_0$. This equation can be non-dimensionalized to

$$\frac{d^2X}{dT^2} = -X - F\left(\frac{dX}{dT} - V\right) \tag{3}$$

Suppose that $V = 1$. Also, suppose that the friction force as a function of relative speed has the following form

$$F(x) = \begin{cases} 1 + ax & : \quad x > 0 \\ -1 + ax & : \quad x < 0 \end{cases} \tag{4}$$

a. Perhaps the simplest model of friction is Coulomb friction, which is Eq. 4 with $a = 0$. Show that linearization predicts that the unique fixed point, $X = -F(-1) = 1$, $dX/dT = 0$, is a center and explain why this is, in fact, a true center.

b. Show that, as a varies, the fixed point goes from a stable to an unstable spiral (assuming $|a| < 2$).

c. It turns out that when the fixed point becomes unstable, a limit cycle appears. This is a Hopf bifurcation. Is it a subcritical, supercritical or degenerate Hopf? Briefly (in a sentence or two) explain.

Problem 3 Define a functional $J : X \to \mathbb{R}$ by

$$J(u) = \int_0^{\pi/4} \left\{ \frac{1}{2}(u')^2 + \frac{1}{2}u^4 + u^2 \right\} dx$$
$$X = \left\{ u \in C^2([0,\pi/4]) : u(0) = 0,\ u(\pi/4) = 1 \right\}$$

a. What is the Euler–Lagrange equation for J?

b. Find the function $u \in X$ that minimizes J.

HINT: It turns out that $u'(0) = 1$, which may be helpful in evaluating the constants of integration.

Problem 4 Consider the boundary value problem (BVP)

$$u'' + u = f(x) \qquad 0 < x < 2\pi$$
$$u(0) = 0, \qquad u(2\pi) = 0,$$

for $u \in C^2([0,2\pi])$, where $f \in C([0,2\pi])$ is a given function.

a. Show that a necessary condition for the BVP to have a solution is that

$$\int_0^{2\pi} f(x) \sin x \, dx = 0.$$

b. If a solution of the BVP exists, show that there is a unique solution u such that

$$\int_0^{2\pi} u(x) \sin x \, dx = 0.$$

c. Write down the set of equations satisfied by the generalized Green's function $G(x,\xi)$ for this BVP. (You don't have to solve for G.)

d. Write down the BVP and orthogonality condition that are satisfied by the function

$$u(x) = \int_0^{2\pi} G(x,\xi) f(\xi) \, d\xi.$$

Problem 5 In the relativistic mechanics of planetary motion around the Sun, one comes across the problem

$$\frac{d^2u}{d\theta^2} + u = \alpha\left(1 + \epsilon u^2\right),$$

where $\alpha > 0$. Here, $u = 1/r$, where r is the normalized radial distance of the planet from the sun, and θ is the angular coordinate in the orbital plane. Find a first-term approximation of the solution u that is valid for large θ for small ϵ that satisfies the initial conditions

$$u(0) = 1$$
$$u'(0) = 0.$$

Problem 6 Find the leading order composite expansion for small ϵ for the problem

$$\epsilon^2 y'' + \epsilon\frac{3}{2}xy' - y = -x, \quad \text{for } 0 < x < 1$$
$$y(0) = 1,$$
$$y(1) = 2.$$

Applied Math Prelim Examination (Spring 2015)

Instructions.

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State all results and theorems that you are using.
2. Use separate sheets for the solution of each problem.

1. Consider the following two sets of coupled ODEs.

Set 1 (Eqs. 1)

$$\begin{aligned}\frac{dx}{dt} &= -y - x(x^2 + y^2) \\ \frac{dy}{dt} &= x - y(x^2 + y^2) \end{aligned} \tag{1}$$

Set 2 (Eqs. 2)

$$\begin{aligned}\frac{dx}{dt} &= -y + xy^2 \\ \frac{dy}{dt} &= x - x^2 y \end{aligned} \tag{2}$$

- Show that, for both sets of ODEs, linear stability predicts that the fixed point $(x = 0, y = 0)$ is a center.
- For one set of ODEs, the fixed point $(x = 0, y = 0)$ is, in fact, a stable spiral. Which one? Is it possible for the linearized equations to correctly predict the stability of the fixed-point? Why or why not?
- For one set of ODEs, the fixed point $(x = 0, y = 0)$ is, in fact, a center. Which one? Show that, for this set of ODEs, closed orbits exist.

2. Consider the following ODE

$$\frac{dx}{dt} = x(x - a) + b \tag{3}$$

- Sketch bifurcation diagrams for 1) $b = 0$; 2) $b = \varepsilon$; and 3) $b = -\varepsilon$, where ε is a small, positive constant.

  (On your bifurcation diagram, indicate stable fixed points with a solid line, unstable fixed points with a dashed line and label all bifurcations).
- Sketch a stability diagram.

  (Recall that a stability diagram will have a and b as axes, and will indicate regions where there are different numbers of fixed points).

3. Consider waves in a resistant medium that satisfy the problem

$$\begin{aligned} u_{tt} =& u_{xx} - \mu u_t, \text{ for } 0 < x < \pi \\ u_x(0,t) = 0,\ \ & u_x(\pi,t) + u(\pi,t) = 0, \\ u(x,0) =& \phi(x),\ \ u_t(x,0) = \psi(x), \end{aligned}$$

where $\mu > 0$ is a constant. Write down the Fourier series expansion of the solution.

4.

(a) Show that

$$\int_a^x \int_a^s f(t)dtds = \int_a^x (x-t)f(t)dt.$$

(b) Express the linear second order ODE,

$$\begin{aligned} & y'' + \alpha y' + c^2 y = 0. \\ & y(0) = 0,\ y'(0) = 1, \end{aligned}$$

as an integral equation of the form

$$y(x) = h(x) + \int_0^x K(x,t)y(t)dt.$$

Determine the functions $h(x)$ and $K(x,t)$?

(c) What is the asymptotic behavior of y (as $x \to \infty$) as a function of the sign of α?

5. Find the the first two terms in the asymptotic approximation of the integral

$$\int_0^1 e^{x[t(1-t^2)]}\, dt.$$

in the following two limits: (a) $x \to -\infty$ and (b) $x \to \infty$. (Hint. Use two different methods to study the cases (a) and (b).)

6. The equation of motion for a pendulum of length L is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0\,,$$

where $\theta(t)$ is the angle measured from the downward vertical direction, and g is the acceleration of gravity. For small initial data,

$$\theta(0) = \epsilon \ll 1, \quad \frac{d\theta}{dt}(0) = 0$$

use the method of multiple scales to calculate the first two terms in the asymptotic expansion (in ϵ) of the frequency of the pendulum. You will need to introduce the slow time scale $T = \epsilon^2 t$.

Applied Mathematics Preliminary Exam (Fall 2015)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the system

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x^5.$$

(a) Is the equilibrium $(x, y) = (0, 0)$: (i) linearly stable; (ii) linearly asymptotically stable; (iii) hyperbolic? What do your answers imply about the nonlinear stability of the equilibrium?

(b) Find a Liapunov function for the system of the form

$$V(x, y) = Ax^6 + By^2.$$

What can you conclude about the nonlinear stability of $(0, 0)$ from the Liapunov function?

Problem 2 Consider the discrete dynamical system with iterates x_n given by the map

$$x_{n+1} = -\mu x_n - x_n^3,$$

where μ is a real parameter.

(a) Find the fixed points of the system as a function of μ and determine their linearized stability.

(b) What kind of bifurcation occurs at $x_n = 0$ as μ increases through $\mu = 1$?

(c) If x_n is small and $\mu = 1 + \epsilon$ is close to 1, show that

$$x_{n+2} \approx (1 + 2\epsilon)x_n + 2x_n^3,$$

after neglecting smaller terms. Determine whether the bifurcation in (b) is subcritical or supercritical.

Problem 3 Find among all continuous curves of length ℓ in the upper half-plane of $\mathbb{R}^2$ passing through $(-a, 0)$ and $(a, 0)$, the one that, together with the interval $[-a, a]$, encloses the largest area. Then, compute the maximum area too.

[Hint: You may want to use the *symmetry* of the problem to your advantage! Also, note that the length of the curve ℓ does not include the length of the interval $2a$ on the horizontal axis.]

Problem 4 Consider a simple rectangular domain $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < a, 0 < y < b\}$ with $a > b$, and the simple heat equation with the following initial and boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} & \text{for } (x, y, t) \in \Omega \times [0, \infty); \\ \dfrac{\partial u}{\partial x}(0, y, t) = \dfrac{\partial u}{\partial x}(a, y, t) = 0, & \text{on } 0 \le y \le b, t \in [0, \infty); \\ \dfrac{\partial u}{\partial y}(x, 0, t) = \dfrac{\partial u}{\partial y}(x, b, t) = 0, & \text{on } 0 \le x \le a, t \in [0, \infty); \\ u(x, y, 0) = f(x, y), & \text{on } (x, y) \in \Omega. \end{cases}$$

(a) Write down the general solution of this problem as a double Fourier series. [Hint: Use the separation of variables.]

(b) Identify the spatial modes (i.e., Fourier basis functions involving only (x, y) variables, not t) corresponding to the three lowest frequencies.

(c) Determine the solution of the above initial and boundary value problem in the case of $f(x, y) \equiv c$ = a real-valued constant.

Problem 5 Consider the following *regular* Sturm-Liouville problem (RSLP):

$$\begin{cases} f'' + \omega^2 f = g & 0 \le x \le 1; \\ f'(0) = 0 = f'(1), \end{cases}$$

where $\omega > 0$ is *not* an integer multiple of π.

(a) Find the Green's function for this RSLP.

(b) What happens if we try this with $\omega = 0$?

Problem 6 Find a one-term approximation, valid to order ε, of the solution to the following differential equation

$$\varepsilon \frac{d^2 y}{dx^2} + y\left(\frac{dy}{dx} + 3\right) = 0$$

for $0 < x < 1$, with boundary conditions $y(0) = -1$ and $y(1) = 1$.

It might be useful to know that

$$\int \frac{1}{-0.5x^2 + a} dx = \sqrt{\frac{2}{a}} \tanh^{-1}\left(x\sqrt{\frac{1}{2a}}\right) + b$$

where a is a positive constant and b is a constant.

It also might be useful to know that tanh is an odd function and that $\lim_{x\to\infty} \tanh(x) = 1$ and $\lim_{x\to-\infty} \tanh(x) = -1$.

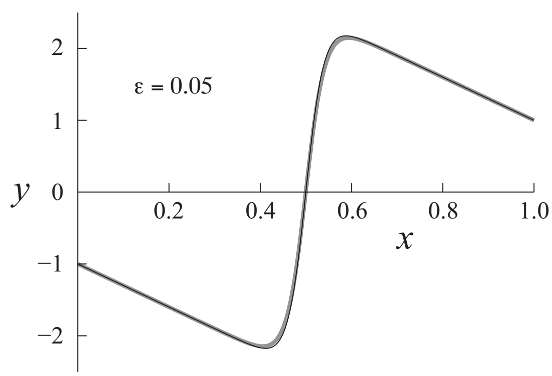


Figure 1: This is a numerical solution of the equation with $\varepsilon = 0.05$ (gray), plotted with my solution for the one-term approximation (black).

GGAM 207 Preliminary Exam (Spring 2014)

Instructions

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1 Consider

$$\begin{aligned} \frac{dx}{dt} &= x(a-x-y) \\ \frac{dy}{dt} &= (y-2a)(x-y) \end{aligned}$$

(a) Find all the equilibrium points.

(b) Find the linear (in)stability of each equilibrium point as a function of a.

(c) Sketch the phase portrait for representative values of a.

(d) Sketch the bifurcation diagram in the (a, x)-plane.

Problem 2 Find the shortest distance between two points (a, b) and (c, d) in $\mathbb{R}^2$ using the Calculus of Variations.

[Hint: Consider a curve $(x(t), y(t))$, $0 \le t \le 1$ with $(x(0), y(0)) = (a, b)$ and $(x(1), y(1)) = (c, d)$.]

Problem 3 Consider the following *regular* Sturm-Liouville problem:

$$\begin{cases} (xf')' + \lambda x^{-1} f = 0 & 1 \le x \le \mathrm{e}; \\ f(1) = f(\mathrm{e}) = 0. \end{cases}$$

(a) Find the eigenvalues and *normalized* eigenfunctions of the above RSL problem. [Hint: Convert this into a simpler RSL problem using the change of variable of x.]

(b) Expand the function $g(x) \equiv 1$ in terms of these eigenfunctions.

Problem 4 Find the leading order uniform approximation to the solution $y(x)$ of

$$\epsilon y'' - (1+x)^2 y' + y = 0, \quad y(0) = 1, \quad y(1) = 0$$

in the limit $\epsilon \downarrow 0^+$. [Hint: boundary layer theory.]

Problem 5 Use the method of stationary phase to find the leading order approximation, as $x \to \infty$, of

$$\int_0^1 e^{ixt^2}\,dt.$$

Problem 6 A wave h of single frequency ω in a medium of variable speed $c(x) > 0$ satisfies

$$\frac{d}{dx}\left[c^2(x)\frac{dh}{dx}\right] + \omega^2 h = 0.$$

(a) What is the condition under which the WKB method produces a good approximation? Under this condition, compute the WKB approximation of h up to second order.

(b) Suppose $c(x) \to c_\pm$ as $x \to \pm\infty$. What are the wavelengths as $x \to \pm\infty$? Let $h_+ = \lim_{x\to\infty} |h(x)|$ and $h_- = \lim_{x\to-\infty} |h(x)|$. With the WKB approximation, determine h_+ in terms of $c_\pm$ and h_-.

Applied Math Prelim Exam (Fall 2014)

Instructions.

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State all results and theorems that you are using.
2. Use separate sheets for the solution of each problem.

1. Consider the mechanical system pictured below. A particle attached to a spring, of rest length 1, slides along a rigid rod. The rigid rod is situated a distance h and at an angle θ from the surface to which the spring is attached.

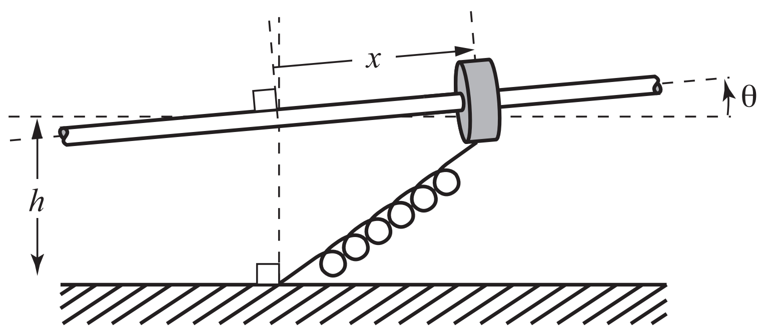


FIGURE 1. A mechanical system.

Assuming that damping is large, the differential equation that governs the particle's position is of the form

$$\frac{dx}{dt} = \left(\frac{1}{\sqrt{h^2 + 2xh\sin(\theta) + x^2}} - 1 \right) (x + h\sin(\theta)) \; .$$

There are two parameters in the equation, θ and h. Sketch a bifurcation diagram in h for the case where $\theta = 0$. Then, sketch a bifurcation diagram in θ for the case where $h = 1 + \varepsilon$ (where ε is an arbitrarily small, but non-zero, positive number). Finally, sketch a stability diagram in h and θ, and find an equation for the boundaries between the phases. In all of your answers, assume that $h \geq 0$ and $-\pi/2 < \theta < \pi/2$.

2. A generic conservative, one degree-of-freedom mechanical system obeys the following differential equations

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -g(x_1)$$

Suppose that there is a local, isolated minimum of the potential energy function $V(x_1) = \int_0^{x_1} g(x_1)$ at x_1^*. Show that this minimum at x_1^* corresponds to a stable equilibrium at $x_1 = x_1^*$, $x_2 = 0$. You may assume that g is C^1.

Note. Recall that a fixed point $\bar{x}$ is asymptotically stable if all nearby trajectories converge to $\bar{x}$ as time $t \to \infty$; it is Lyapanov stable if nearby trajectories remain close to $\bar{x}$ for all time. In this problem, by "stability," we refer to either type.

3. Find the path between (x_1, y_1) and (x_2, y_2) which a particle sliding without friction and under constant gravitational acceleration will traverse in the shortest time. You may assume that the particle is released from (x_1, y_1) at rest and hence conservation of energy implies that

$$\frac{1}{2}mv^2 + mgy = mgy_1.$$

4. Determine the Green's function associated with the BVP

$$x^2y'' - xy' - 3y = x - 3, \quad y(1) = 0, y(2) = 0,$$

and give a solution to the BVP.

5. Find the leading order approximations in the limits $x \to \infty$ and $x \to -\infty$ of

$$\int_0^\pi e^{x \sin(t)}\, dt\,.$$

6. For $0 < \epsilon \ll 1$ and $k(\epsilon x) > 0$, $\forall x \in \mathbb{R}$, with $k(\epsilon x) \sim \mathcal{O}(1)$, consider the following two second order ODEs:

$$\frac{d}{dx}\left[\frac{1}{k(\epsilon x)^2}\frac{dh_1}{dx}\right] + h_1 = 0 \tag{1}$$

$$\frac{1}{k(\epsilon x)^2}\frac{d^2h_2}{dx^2} + h_2 = 0\,. \tag{2}$$

Equation (1) describes the amplitude $h_1(x)$ of a wave in a medium with varying wave speed, while equation (2) describes the amplitude $h_2(x)$ of a harmonic oscillator with varying frequency.

Compute the WKB approximation (up to $\mathcal{O}(\epsilon)$, i.e., two terms in the asymptotic expansion) for both equations (1) and (2). Furthermore, assuming that

$$\lim_{x\to-\infty} k \to k_- \quad \text{and} \quad \lim_{x\to\infty} k \to k_+\,,$$

and that

$$\lim_{x\to-\infty} |h_1(x)|^2 = 1 \text{ and } \lim_{x\to-\infty} |h_2(x)|^2 = 1,$$

determine the limits

$$\lim_{x\to\infty} |h_1(x)|^2 \text{ and } \lim_{x\to\infty} |h_2(x)|^2\,.$$

Spring 2013: PhD Applied Math Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Consider the 2×2 system of ODEs, where a, b are real constants,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} + a(x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix} + b(x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix}.$$

(a) Linearize the system at the origin. Classify the equilibrium of the linearized system and determine its linearized stability.

(b) Write the system in polar coordinates, sketch the phase plane, and determine how the nonlinear stability of the origin depends on (a, b).

Problem 2. Consider the following initial-value problem for an infinite-dimensional system of ODEs for real-valued functions $\{x_1(t), x_2(t), x_3(t), \ldots\}$

$$\frac{dx_n}{dt} = n^2 x_n^3, \qquad x_n(0) = c_n, \qquad n = 1, 2, 3, \ldots.$$

(a) Solve for $x_n(t)$.

(b) If $\sum_{n=1}^{\infty} n^2 c_n^2 \leq 1$, show that a solution exists in some time interval $|t| < T$, and give an estimate for the minimal existence time $T > 0$.

(c) If $\sum_{n=1}^{\infty} c_n^2 \leq 1$, show that a solution need not exist in any interval $|t| < T$, however small one chooses $T > 0$,.

Problem 3. Let

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

where p, q are smooth, real-valued functions on on $a \leq x \leq b$ and $p(x) > 0$.

(a) Define the Green's function $G(x, \xi)$ for the regular Sturm-Liouville problem $Lu = f$ for $a < x < b$, with $u(a) = 0$, $u'(b) = 0$.

(b) Show that G is symmetric i.e. $G(x, \xi) = G(\xi, x)$, and give a physical interpretation of this symmetry.

Problem 4. Let $X = \{u \in C^2([1,2]) : u(1) = 0,\ u(2) = 1\}$ and define the functional $J : X \to \mathbb{R}$ by

$$J(u) = \int_1^2 \frac{\sqrt{1 + (u')^2}}{x}\, dx.$$

(a) Write down the Euler-Lagrange equation associated with J.

(b) Solve the Euler-Lagrange equation to find the minimizer of J on X.

Problem 5. Consider a vibrating string that is initially at rest and is subject to a spatially dependent, time-periodic external force with frequency ω. Suppose that the displacement $u(x,t)$ satisfies

$$\begin{aligned}
&u_{tt} - c^2 u_{xx} = A \sin\left(\frac{\pi x}{L}\right) \sin(\omega t) \qquad 0 < x < L,\ 0 < t,\\
&u(0,t) = 0, \qquad u(L,t) = 0, \qquad 0 \le t\\
&u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad 0 \le x \le L.
\end{aligned}$$

where $A \neq 0$ is the amplitude of the external force.

(a) Solve this IBVP for $u(x,t)$.

(b) For what values of ω is the solution also periodic in time?

Problem 6. Let $\nu > 0$ and $-\infty < U < \infty$ be constants, and consider the following PDE with boundary conditions at $x = \pm\infty$:

$$\begin{aligned}
&u_t + u u_x = \nu u_{xx}, \qquad -\infty < x < \infty,\ 0 < t,\\
&u(x,t) \to U \text{ as } x \to -\infty, \qquad u(x,t) \to 0 \text{ as } x \to \infty.
\end{aligned} \tag{1}$$

(a) If x, t, and u have dimensions of length, time, and velocity, respectively, show that this problem is dimensionally consistent. Determine the dimensions of ν and U, and use ν and U to nondimensionalize the problem.

(b) Consider traveling wave solutions $u = u(x - ct)$ of (1). What can you say using dimensional analysis about the speed c and a typical width L of a traveling wave?

(c) Find a first-order ODE for the traveling wave profile $u = u(z)$, where $z = x - ct$. Show that traveling waves exist if $U > 0$ but not if $U < 0$, and verify the results of the dimensional analysis.

GGAM Prelim Questions - Fall 2013

Instructions:

- **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.**
- **Use separate sheets for the solution of each problem.**

1. Consider the second order ODE which describes the height, $h(x)$, of a wave in a medium with varying wave speed

$$\frac{d^2h}{dx^2} + k^2(x)h = 0$$

where $k(x)$ is the local wavenumber

$$k(x) = k_1 + (k_2 - k_1)\tanh(x/L), \quad 0 < k_1 < k_2.$$

   (a) Non-dimensionalize the system by using L to measure length.

   (b) For $L \gg 1$, write down the first two leading order (i.e. eikonal and transport) equations for the WKB approximation of $h(x)$ (Do this by considering a general form of $k(x)$ - you need not substitute the particular $k(x)$).

   (c) Suppose the wave profile is asymptotically

$$h(x) = Ae^{ik_1x} \quad \text{for} \quad x \to -\infty.$$

   Solve the WKB equation(s) to determine the asymptotic profile $|h(x)|$ for $x \to \infty$.

2. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + \sqrt{x}\,y' + y = 0 \quad \text{in} \quad 0 \le x \le 1$$

with boundary conditions $y(0) = 0$, and $y(1) = 1$. Find the matched asymptotic (inner and outer) solutions.

3. The small, centrally symmetric vibrations of a stretched uniform circular membrane, fixed round its perimeter, are approximately described by the equations

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right), & 0 \le r \le R,\ t \ge 0; \\ u(R,t) = 0, & t \ge 0. \end{cases}$$

Here R is the radius of the membrane, $u(r,t)$ is the transverse displacement of a point distant r from the center of the membrane at time t, and a is a positive constant.

   (a) Separate variables to obtain a singular Sturm-Liouville system.

   (b) Find the eigenvalues of this system in terms of the zeros of the Bessel function J_0, and write down the corresponding eigenfunctions.

4. Consider the following *regular* Sturm-Liouville (RSL) problem:

$$\begin{cases} f'' + \lambda f = 0 & 0 \le x \le \ell; \\ f'(0) = f(\ell) = 0. \end{cases}$$

   (a) Find the eigenvalues and *normalized* eigenfunctions of the above RSL.

   (b) Let $S = \text{span}\{\phi_1, \phi_2\}$, the subspace of $L^2(0,\ell)$ consisting of all possible linear combinations of the first two eigenfunctions ϕ_1, ϕ_2 of the above RSL. Find the best linear approximation in S to the function $g(x) = \ell^2 - x^2$ in the L^2 sense.

5. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= ax + y - xf(x^2 + y^2) \\ \frac{dy}{dt} &= -x + ay - yf(x^2 + y^2) \end{aligned}$$

   where a is real, f is continuous, $f(0) = 0$ and $f(u) \ge u^{1/2}$.

   (a) Show that the origin is the only equilibrium point and determine its linear stability.

   (b) Using the Poincare-Bendixson theorem, show that there exists a stable limit cycle if $a > 0$.

   (c) Consider the special case with $f(u) = u^{1/2}$ for all $r \ge 0$ with $a > 0$. Find the limit cycle explicitly.

6. For the solar system, Einstein's General Relativity can be viewed as a small perturbation to the regular Newtonian theory of gravity. The orbit of a planet going around the sun can be described in terms of the polar coordinates by $r(\theta)$ where r is the distance from the planet to the center of mass of the system and θ is the angle of the planet in its orbit. In General Relativity, $r(\theta)$ is approximately governed by the equation

$$\frac{d^2 r}{d\theta^2} + r = \frac{1}{L} + \epsilon L r^2, \quad 0 < \epsilon \ll 1$$

   where L is related to the angular momentum of the planet and ϵ is a small positive parameter representing the deviation from the Newtonian theory. When $\epsilon = 0$, this is the equation for Newtonian gravity.

   (a) Find the equilibrium points and classify their stability for $\epsilon > 0$.

   (b) Find the limits of the equilibrium points as $\epsilon \to 0$.

   (c) For $\epsilon > 0$ sketch the phase portrait (in the half plane $r \ge 0$) and identify the region where there are periodic solutions in θ.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
March 29, 2012

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)
For $u \in L^1(0,\infty)$, consider the integral

$$v(x) = \int_0^\infty \frac{u(y)}{x+y}\,\mathrm{d}y$$

defined for $x > 0$. Show that $v(x)$ is infinitely differentiable away from the origin. Prove that $v' \in L^1(\epsilon,\infty)$ for any $\epsilon > 0$. Explain what happens in the limit as $\epsilon \to 0$.

Problem 2: (10 points)
Let $X \subset L^2(0,2\pi)$ be the set of all functions $u(x)$ such that

$$u(x) = \lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{ikx} \text{ in } L^2\text{-norm, with } |a_k| \le (1+|k|)^{-1}.$$

Prove that X is compact in $L^2(0,2\pi)$.

Problem 3: (10 points)
For $\epsilon > 0$, we set

$$\eta_\epsilon(x) = \frac{1}{\pi}\sin\left(\frac{\epsilon\pi x}{x^2+\epsilon^2}\right)\frac{\epsilon}{x^2+\epsilon^2},$$

and define the convolution for $u \in L^2(\mathbb{R})$:

$$\eta_\epsilon * u(x) = \int_{\mathbb{R}} \eta_\epsilon(x-y)u(y)\,\mathrm{d}y.$$

For $\epsilon > 0$, prove that $\sqrt{\epsilon}(\eta_\epsilon * u)(x)$ is bounded as a function of x and ϵ, and that $\eta_\epsilon * u$ converges strongly in $L^2(\mathbb{R})$ as $\epsilon \to 0$. What is the limit?

Problem 4: (10 points)
Let $u_n : [0,1] \to [0,\infty)$ denote a sequence of measurable functions satisfying

$$\sup_n \int_0^1 u_n(x)\log(2+u_n(x))\,\mathrm{d}x < \infty.$$

If $u_n(x) \to u(x)$ almost everywhere, show that $u \in L^1(0,1)$ and that $u_n \to u$ in L^1 strongly.

(**Hint.** One possible strategy is Egoroff's Theorem.)

Problem 5: (10 points)
Let $u : [0,1] \to \mathbb{R}$ be absolutely continuous, satisfy $u(0) = 0$, and

$$\int_0^1 |u'(x)|^2\,\mathrm{d}x < \infty.$$

Prove that

$$\lim_{x\to 0^+} \frac{u(x)}{x^{\frac{1}{2}}}$$

exists and determine the value of this limit.

Problem 6: (10 points)
Consider on $\mathbb{R}^2$ the distribution defined by the locally integrable function

$$E(x,t) = \begin{cases} \frac{1}{2} & \text{if } t-|x|>0 \\ 0 & \text{if } t-|x|<0 \end{cases}.$$

Compute the distributional derivative

$$\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2}.$$

Problem 7: (10 points)

Consider

$$\dot{x} = y + ax(1 - 2b - x^2 - y^2) \qquad \dot{y} = -x + ay(1 - x^2 - y^2)$$

with $0 < a \leq 1, 0 \leq b < \frac{1}{2}$; prove that there is at least one limit cycle and calculate the period $T(a, b)$ (*i.e.*, write it as an integral).

Problem 8: (10 points)

Consider the system

$$\dot{x} = y - 2x \qquad \dot{y} = \mu + x^2 - y\,.$$

(a) Sketch the nullclines of the system for different values of μ in order to find and classify the bifurcation that occurs at $\mu = \mu_c$.

(b) Classify the fixed points and sketch the phase portrait for μ slightly smaller than μ_c.

(c) For which values of μ the system admits a stable spiral?

Fall 2012: PhD Applied Math Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1. Consider the one-dimensional discrete dynamical system

$$x_{n+1} = \mu e^{x_n}, \qquad n = 0, 1, 2, \ldots$$

where $x_n \in \mathbb{R}$ and μ is a real parameter.
(a) Describe qualitatively how the the fixed points of the system change as μ increases from $-\infty$ to ∞ and determine their stability.
(b) What types of bifurcation occur when there is a change in stability of the fixed points?

Problem 2. Consider the 3×3 system of ODEs

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1, \qquad \dot{x}_3 = 1 - \left(x_1^2 + x_2^2\right)$$

(a) Show that trajectories of the system in phase space $\{(x_1, x_2, x_3) \in \mathbb{R}^3\}$ lie on the cylinders

$$x_1^2 + x_2^2 = c^2 \tag{1}$$

where $c \geq 0$ is a constant.
(b) Sketch the trajectories on the cylinder (1) for: (i) $c = 0$; (ii) $0 < c < 1$; (iii) $c = 1$; (iv) $c > 1$.
(c) Does the system have any equilibria? Does it have periodic solutions? Why doesn't your answer contradict the Poincaré- Bendixson theorem?

Problem 3. Compute the Green's function for the boundary value problem

$$\begin{aligned} &u'' + u = f(x) \qquad\qquad 0 < x < 1, \\ &u'(0) = 0, \qquad u(1) = 0, \end{aligned}$$

and write out the Green's function representation of the solution.

Problem 4. Suppose $u(x)$ satisfies the following boundary value problem on $[-1, 1]$

$$\begin{aligned} Lu &= f(x) \qquad -1 < x < 1, \\ u(-1) &= 0, \qquad u(1) = 0. \end{aligned} \tag{2}$$

where $f : [-1, 1] \to \mathbb{R}$ is a given smooth function and

$$Lu = u'' + xu' + 3u.$$

(a) Find the formal adjoint L^* of L and the adjoint boundary conditions.
(b) Verify that $v(x) = 1-x^2$ is a solution of the homogeneous adjoint problem and derive a necessary condition that $f(x)$ must satisfy if (2) is solvable.

Problem 5. (a) Suppose that $0 < \epsilon \ll 1$ is a small positive parameter. Use the method of matched asymptotic expansions to construct leading order approximations

$$x = x_0 + O(\epsilon), \qquad y = y_0 + O(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

of the solution $x(t; \epsilon)$, $y(t; \epsilon)$ of the initial value problem

$$\dot{x} = -xy, \qquad \epsilon\dot{y} = x^2 - y, \qquad x(0) = x_0, \quad y(0) = y_0$$

that are valid for times of the order ϵ and times of the order 1.
(b) Sketch the phase plane of this system. How do solutions behave as $t \to +\infty$?

Problem 6. (a) Find all separable solutions $u(x, t) = F(x)G(t)$ of the wave equation on the interval $0 < x < 1$ subject to homogeneous Dirichlet boundary conditions:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad 0 < x < 1 \\ u(0, t) &= 0, \qquad u(1, t) = 0. \end{aligned}$$

(b) Consider a Dirichlet problem for the wave equation on a rectangle of sides length 1 and $T > 0$,

$$\begin{aligned} &u_{tt} - u_{xx} = 0 & & & &0 < x < 1, \quad 0 < t < T, \\ &u(x, 0) = f(x), & &u(x, T) = g(x) & &0 \le x \le 1, \\ &u(0, t) = h(t), & &u(1, t) = k(t) & &0 \le t \le T, \end{aligned}$$

where $f, g : [0, 1] \to \mathbb{R}$ and $h, k : [0, T] \to \mathbb{R}$ are given functions. Suppose that this problem has a solution. Show that solutions are unique if T is irrational and non-unique if T is rational.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
March 24, 2011

Instructions:

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = n\mathrm{e}^{-nx}$. Prove that $f_n \not\rightharpoonup f$ weakly in $L^1(\Omega)$, i.e., the sequence f_n does not converge in the weak topology of $L^1(\Omega)$.
(**Hint**: Prove by contradiction.)

Problem 2: (10 points)

Let $\Omega = (0,1)$, and consider the linear operator $A = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ acting on the Sobolev space of functions X where

$$X = \left\{u \in H^2(\Omega) \mid u(0) = 0\,, u(1) = 0\right\}\,,$$

and where

$$H^2(\Omega) = \left\{u \in L^2(\Omega) \;\middle|\; \frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(\Omega)\,, \frac{\mathrm{d}^2u}{\mathrm{d}x^2} \in L^2(\Omega)\right\}\,.$$

Find all of the eigenfunctions of A belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) \mid \alpha \in \mathbb{R}\}\,,$$

as well as their corresponding eigenvalues.

Problem 3: (10 points)

Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and set

$$v(x) = (1 + |\log x|)^{-1}.$$

Show that $v \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $\dfrac{v}{x} \notin L^1(\Omega)$. (This shows the failure of Hardy's inequality in L^1.) Note that $W^{1,1}(\Omega) = \left\{ u \in L^1(\Omega) \;\middle|\; \dfrac{\mathrm{d}u}{\mathrm{d}x} \in L^1(\Omega) \right\}$, where $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ denotes the weak derivative.

Problem 4: (10 points)

Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Show that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n \tau_n \delta \text{ in } \mathscr{D}', \tag{1}$$

that is, that equality in equation (1) holds in the sense of distributions, and relate b_n to the coefficients of the Fourier series. Note that δ denotes the Dirac distribution and τ_y is the translation operator, given by $\tau_y f(x) = f(x+y)$.

(**Hint:** Write $f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}$ with convergence in $L^2(0,2\pi)$ and where the coefficients $c_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nx} f(x)\,\mathrm{d}x$.)

Problem 5: (10 points)

Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Given $\epsilon > 0$, prove that for $N < \infty$ there is a finite Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] \tag{2}$$

such that

$$|\phi(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R}.$$

This shows that the space of real-valued trigonometric polynomials on $\mathbb{R}$ (functions which can be expressed as in (2)) are *uniformly* dense in the space of periodic continuous function on $\mathbb{R}$ with period 2π.

(**Hint:** The Stone-Weierstrass theorem states that if X is compact in $\mathbb{R}^d$, $d \in \mathbb{N}$, then the algebra of all real-valued polynomials on X (with coordinates $(x_1, x_2, ..., x_d)$) is dense in $C(X)$.)

Problem 6: (10 points)

For $\alpha \in (0,1]$, the space of Hölder continuous functions on the interval $[0,1]$ is defined as

$$C^{0,\alpha}([0,1]) = \{u \in C([0,1]) \,:\, |u(x) - u(y)| \leq C|x-y|^{\alpha}\,, x, y \in [0,1]\}\,,$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x \in [0,1]} |u(x)| + \sup_{x,y \in [0,1]} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}}\,.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) \,:\, \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.

(**Hint:** The Arzela-Ascoli theorem states that if a family of continuous functions U is equicontinuous and uniformly bounded on $[0,1]$, then each sequence u_n in U has a uniformly convergent subsequence. Recall that U is uniformly bounded on $[0,1]$ if there exists $M > 0$ such that $|u(x)| < M$ for all $x \in [0,1]$ and all $u \in U$. Further, recall that U is equicontinuous at $x \in [0,1]$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for all $|x - y| < \delta$ and every $u \in U$.)

Problem 7: (10 points)

Consider the system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x(y+1)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - x^2 - y^2$$

(a) Show that $(x, y) = (0, 1)$ and $(0, -1)$ are fixed point of the system. Linearize the system about the fixed points $(0, 1)$ and $(0, -1)$ and use linearized system to classify the fixed points.

(b) Sketch the phase portrait of the full system and re-classify the fixed points.

Problem 8: (10 points)

Consider the system describing a particle mass moving in a double-well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, i.e.,

$$\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} = x - x^3.$$

(a) Show that the energy $E(x, \dot{x}) = \dfrac{\dot{x}^2}{2} + V(x)$ is a conserved quantity for this system, i.e. $E(x, \dot{x})$ is constant along trajectories.

(b) Sketch the $x, \dot{x}$-phase portrait. Classify the fixed points of the system $(0, 0)$ and $(\pm 1, 0)$.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 20, 2011

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)
Let (X,d) be a metric space and let (x_n) be a sequence in X. For the purpose of this problem adopt the following definition: $x \in X$ is called a *cluster point* of (x_n) iff there exists a subsequence $(x_{n_k})_{k\geq 0}$ such that $\lim_k x_{n_k} = x$.

(a) Let $(a_n)_{n\geq 0}$ be a sequence of distinct points in X. Construct a sequence $(x_n)_{n\geq 0}$ in X such that for all $k = 0,1,2,\ldots$, a_k is a cluster point of (x_n).

(b) Can a sequence (x_n) in a metric space have an *uncountable* number of cluster points? Prove your answer. (If you answer yes, give an example with proof. If you answer no, prove that such a sequence cannot exists). You may use without proof that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

Problem 2: (10 points)
Let X be a real Banach space and X^* its Banach space dual. For any bounded linear operator $T \in \mathscr{B}(X)$, and $\phi \in X^*$, define the functional $T^*\phi$ by

$$T^*\phi(x) = \phi(Tx), \quad \text{for all } x \in X.$$

(a) Prove that T^* is a bounded operator on X^* with $\|T^*\| \leq \|T\|$.

(b) Suppose $0 \neq \lambda \in \mathbb{R}$ is an eigenvalue of T. Prove that λ is also an eigenvalue of T^*. (**Hint 1:** first prove the result for $\lambda = 1$. **Hint 2:** For $\phi \in X^*$, consider the sequence of Cesàro means $\psi_N = N^{-1}\sum_{n=1}^{N} \phi_n$, of the sequence ϕ_n defined by $\phi_n(x) = \phi(T^n x)$.)

Problem 3: (10 points)

Let $\mathscr{H}$ be a complex Hilbert space and denote by $\mathscr{B}(\mathscr{H})$ the Banach space of all bounded linear transformations (operators) of $\mathscr{H}$ considered with the operator norm.

(a) What does it mean for $A \in \mathscr{B}(\mathscr{H})$ to be *compact*? Give a definition of compactness of an operator A in terms of properties of the image of bounded sets, e.g., the set $\{Ax \mid x \in \mathscr{H}, \|x\| \le 1\}$.

(b) Suppose $\mathscr{H}$ is separable and let $\{e_n\}_{n\ge 0}$ be an orthonormal basis of $\mathscr{H}$. For $n \ge 0$, let P_n denote the orthogonal projection onto the subspace spanned by $e_0, \dots, e_n$. Prove that $A \in \mathscr{B}(\mathscr{H})$ is compact iff the sequence $(P_n A)_{n\ge 0}$ converges to A in norm.

Problem 4: (10 points)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and smooth. Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ and $f_j \rightharpoonup g_1$ weakly in $L^2(\Omega)$ and that $f_j(x) \to g_2(x)$ a.e. in Ω. Show that $g_1 = g_2$ a.e. (**Hint:** Use Egoroff's theorem which states that given our assumptions, for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\lambda(E) < \epsilon$ and $f_j \to g_2$ uniformly on E^c.)

Problem 5: (10 points)

Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$, $u(0) = 0$, but $\dfrac{u}{x} \notin L^1(0,1)$.

Problem 6: (10 points)

Let $H = \left\{ f \in L^2(0,2\pi) \;:\; \displaystyle\int_0^{2\pi} f(x)\,\mathrm{d}x = 0 \right\}$. We define the operator Λ as follows:

$$(\Lambda f)(x) = \int_0^x f(y)\,\mathrm{d}y.$$

(a) Prove that $\Lambda : H \to L^2(0,2\pi)$ is continuous.

(b) Use the Fourier series to show that the following estimate holds:

$$\|\Lambda f\|_{H_0^1(0,2\pi)} \le C \|f\|_{L^2(0,2\pi)},$$

where C denotes a constant which depends only on the domain $(0,2\pi)$. (Recall that $\|u\|^2_{H_0^1(0,2\pi)} = \displaystyle\int_0^{2\pi} \left| \frac{\mathrm{d}u}{\mathrm{d}x}(x) \right|^2 \mathrm{d}x$.)

Problem 7: (10 points)

Consider the system

$$\dot{x} = \mu x + y + \tan x \qquad \dot{y} = x - y\ .$$

(a) Show that a bifurcation occurs at the origin $(x, y) = (0, 0)$, and determine the critical value $\mu = \mu_c$ at which the bifurcation occurs.

(b) Determine the type of bifurcation that occurs at $\mu = \mu_c$. Do this (i) analytically and (ii) graphically (sketch the appropriate phase portraits for μ slightly less than; equal to; and slightly greater than μ_c).

Problem 8: (10 points)

Consider the differential equation

$$\ddot{x} + x - x^3 = 0\ ,$$

with the initial condition $x(0) = \epsilon$, $\dot{x}(0) = 0$, where $\epsilon \ll 1$. Use "two-timing" and perturbation theory to approximate the frequency of oscillation to order ϵ^2.

(a) Make a change of variables so that the differential equation is in the form $\ddot{z} + z + \epsilon h(z, \dot{z}) = 0$, i.e., in a form where ϵ appears naturally in the equation as a perturbation parameter.

(b) Rewrite the equation assuming two times scales, a fast time $\tau = t$ and a slow one $T = \epsilon\tau$, and the solution form $z(t, \epsilon) = z_0(\tau, T) + \epsilon z_1(\tau, T) + O(\epsilon^2)$.

(c) Show that the order 0 (i.e., $O(1)$) solution takes the form

$$z_0(\tau, T) = r(T)\cos(\tau + \phi(T))\ .$$

(d) Use the order 1 (i.e., $O(\epsilon)$) equation to determine the frequency of oscillation to order ϵ^2. (**Hint:** The order 1 (i.e., $O(\epsilon)$) equation contains resonant terms, which would cause the solution to grow without bound as $t \to \infty$. A solution that remains bounded for large τ is obtained by setting the coefficients of the resonant terms to zero. This yields equations that can be used to find the order ϵ^2 correction for the frequency of the oscillation. Note: Be sure to look for "hidden" resonance terms. It may be helpful to use the trig identity $\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)$.)

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
April 1, 2010

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)
Let (X, d) be a complete metric space, $\bar{x} \in X$, and $r > 0$. Set $D := \{x \in X : d(x, \bar{x}) \leq r\}$, and let $f : D \to X$ satisfying

$$d\left(f(x), f(y)\right) \leq k\, d(x, y)$$

for any $x, y \in D$, where $k \in (0, 1)$ is a constant.
Prove that if $d\left(\bar{x}, f(\bar{x})\right) \leq r(1-k)$, then f admits a unique fixed point. (Guidelines: Assume the Banach fixed point theorem, also known as the contraction mapping theorem.)

Problem 2: (10 points)
Give an example of two normed vector spaces, X and Y, and of a sequence of operators, $\{T_n\}_{n=0}^{\infty}$, $T_n \in L(X, Y)$ (where $L(X, Y)$ is the space of the continuous operators from X to Y, with the topology induced by the operator norm) such that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence but it does not converge in $L(X, Y)$. (Notice that Y cannot be a Banach space otherwise $L(X, Y)$ is complete.)

Problem 3: (10 points)
Let (a_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^3$$

converges. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

also converges.

Problem 4: (10 points)

Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable function from the square to the square with a continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

Problem 5: (10 points)

Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$, where $\hat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ix\xi} \, dx$ denotes the Fourier transform of u.

Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \le \epsilon \|u\|_t + C \|u\|_r \quad \forall u \in H^t(\mathbb{R}).$$

Problem 6: (10 points)

Let $f : [0,1] \to \mathbb{R}$. Show that f is continuous if and only if the graph of f is compact in $\mathbb{R}^2$.

Problem 7: (10 points)

The precession of the perihelion of a planet in Einstein's Theory of General Relativity. In our solar system, General Relativity can be viewed as a small perturbation to the regular Newtonian theory of gravity. When studying the problem of planets going around the sun, you get an equation for $u = r^{-1}$ where r is the distance from the planet to the center of mass of the system and θ is the angle of the planet in its orbit. In General Relativity, the equivalent equation is approximately

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{1}{L} + \epsilon L u^2$$

where L is related to the angular momentum and $0 < \epsilon \ll 1$. When $\epsilon = 0$, this is the equation for Newtonian gravity.

(a) What does a circular orbit correspond to in this system?

(b) Find the fixed points and classify their stability. (There is a center and a saddle). Also, expand the location of the center in a Taylor series in ϵ, retaining only the first two terms.

(c) Sketch the phase portrait and identify the region where there are periodic solutions in θ.

(d) When $\epsilon = 0$, what is the period of all of the closed orbits, $r(\theta)$? What does this mean for the shape of the orbits in physical space (i.e., what do the ACTUAL planetary trajectories look like and how is this related to the result you just found?). Sketch one orbit.

(e) Find and approximate expression for the period of nearly circular orbits when $0 < \epsilon \ll 1$. What does this mean for the shape of the orbits in physical space? Sketch two periods of such an orbit.

Problem 8: (10 points)

Consider the system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x(a - x - y)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2a(y - x) + y(x - y).\end{aligned}$$

(a) Find the equilibrium points and for what values of a they exist.

(b) Find the linear stability of each point as a function of a.

(c) Sketch the phase portrait for representative values of a.

(d) Sketch the bifurcation diagram in the (a, x)-plane.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 21, 2010

Instructions:

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)
Let $f(x,y)$ denote a C^1 function on $\mathbb{R}^2$. Suppose that

$$f(0,0)=0.$$

Prove that there exist two functions, $A(x,y)$ and $B(x,y)$, both continuous on $\mathbb{R}^2$ such that

$$f(x,y) = xA(x,y) + yB(x,y) \quad \forall (x,y) \in \mathbb{R}^2$$

(**Hint**: Consider the function $g(t) = f(tx,ty)$ and express $f(x,y)$ in terms of g via the fundamental theorem of calculus .)

Problem 2: (10 points)
The Fourier transform $\mathscr{F}$ of a distribution is defined via the duality relation

$$\langle \mathscr{F} f, \phi \rangle = \langle f, \mathscr{F}^* \phi \rangle$$

for all $\phi \in C_0^\infty(\mathbb{R})$, the smooth compactly-supported test functions on $\mathbb{R}$, where

$$\mathscr{F}^* \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x} \phi(\xi)\,\mathrm{d}\xi .$$

Explicitly compute $\mathscr{F} f$ for the function

$$f(x) = \begin{cases} x, & x>0 \\ 0, & x \le 0 \end{cases} .$$

Problem 3: (10 points)

Let $\{P_n(x)\}_{n=1}^{\infty}$ denote a sequence of polynomials on $\mathbb{R}$ such that

$$P_n \to 0 \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty.$$

Prove that, for n sufficiently large, all P_n are constant polynomials.

Problem 4: (10 points)

For $g \in L^1(\mathbb{R}^3)$, the convolution operator G is defined on $L^2(\mathbb{R}^3)$ by

$$Gf(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} g(x-y) f(y) \, \mathrm{d}y, \quad f \in L^2(\mathbb{R}^3).$$

Prove that the operator G with

$$g(x) = \frac{1}{4\pi} \frac{\mathrm{e}^{-|x|}}{|x|}, \quad x \in \mathbb{R}^3,$$

is a bounded operator on $L^2(\mathbb{R}^3)$, and the operator norm $\|G\|_{op} \le 1$.

Problem 5: (10 points)

Consider the map which associates to each sequence $\{x_n : n \in \mathbb{N}, x_n \in \mathbb{R}\}$ the sequence, $\{(F(\{x_n\}))_m ; m \in \mathbb{N}, (F(\{x_n\}))_m \in \mathbb{R}\}$, defined as follows:

$$\left\{F(\{x_n\})\right\}_m := \frac{x_m}{m} \quad \text{for} \quad m = 1, 2, \ldots$$

1. Determine (with proof) the values of $p \in [1, \infty]$ for which the map $F : l^p \to l^1$ is well-defined and continuous.
2. Next, determine the values of $q \in [1, \infty]$ for which the map $F : l^q \to l^2$ is well-defined and continuous.

Note for $1 \leq p < \infty$, l^p denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, while l^∞ denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sup_{n \in \mathbb{N}} |x_n| < \infty$.

Problem 6: (10 points)

For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof, e.g., by citing a relevant theorem; if false, give a counterexample.

Let $\mathbb{H}$ denote a separable Hilbert space and (x_n) a sequence of $\mathbb{H}$.

(a) If (x_n) is weakly convergent then it is strongly convergent.

(b) If (x_n) is strongly convergent then it is bounded.

(c) If (x_n) is weakly convergent then it is bounded.

(d) If (x_n) is bounded, there exists a strongly convergent subsequence of (x_n).

(e) If (x_n) is bounded, there exists a weakly convergent subsequence of (x_n).

(f) If (x_n) is weakly convergent and T is a bounded linear operator from $\mathbb{H}$ to $\mathbb{R}^d$, for some d, then $T(x_n)$ converges in $\mathbb{R}^d$.

Problem 7: (10 points)

Consider the first order ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta + \alpha x - x^3 = f(x)\,,$$

with $\alpha, \beta \in \mathbb{R}$.

(a) What conditions on $f(x)$ and $f'(x)$ must be satisfied simultaneously at bifurcation points? *Briefly* explain your answers.

(b) Use these conditions to find the curves of bifurcation points in α vs. β parameter space? Sketch the corresponding curves in the α vs. β plane.

(c) Sketch the following bifurcation diagrams. Indicate the stability of the fixed points on the diagrams and classify the bifurcations that occurs. A detailed analytical treatment of the system is not required, but some justification (e.g., graphical arguments) of your answers is required.

(i) Use α as the bifurcation parameter and hold β constant at $\beta = 0$.

(ii) Use α as the bifurcation parameter and hold β constant with $\beta > 0$.

(iii) Use β as the bifurcation parameter and hold α constant with $\alpha > 0$.

Problem 8: (10 points)

Consider the system of ordinary differential equations in polar coordinates

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1-r^2)(4-r^2), \qquad r \geq 0$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2 - r^2.$$

Sketch the phase-portrait. Label all fixed points and limit cycles, and indicate their stability.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
January 2, 2009

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let $1 < p < 2$.

(a) Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.

(b) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$.

Problem 2: (10 points)

(a) State the Weierstrass approximation theorem.

(b) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x)\,\mathrm{d}x = 0$$

for all non-negative integers n. Prove that $f = 0$.

Problem 3: (10 points)

(a) Define strong convergence, $x_n \to x$, and weak convergence, $x_n \rightharpoonup x$, of a sequence (x_n) in a Hilbert space $\mathscr{H}$.

(b) If $x_n \rightharpoonup x$ weakly in $\mathscr{H}$ and $\|x_n\| \to \|x\|$, prove that $x_n \to x$ strongly.

(c) Give an example of a Hilbert space $\mathscr{H}$ and sequence (x_n) in $\mathscr{H}$ such that $x_n \rightharpoonup x$ weakly and

$$\|x\| < \liminf_{n\to\infty} \|x_n\|.$$

Problem 4: (10 points)

Suppose that $T : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator on a complex Hilbert space $\mathscr{H}$ such that

$$T^* = -T, \qquad T^2 = -I$$

and $T \neq \pm \mathrm{i}I$. Define

$$P = \frac{1}{2}(I + \mathrm{i}T), \qquad Q = \frac{1}{2}(I - \mathrm{i}T).$$

(a) Prove that P, Q are orthogonal projections on $\mathscr{H}$.

(b) Determine the spectrum of T, and classify it.

Problem 5: (10 points)

Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : \mathscr{S}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\widehat{(Hf)}(\xi) = \mathrm{i}\,\mathrm{sgn}(\xi)\hat{f}(\xi) = \begin{cases} \mathrm{i}\hat{f}(\xi) & \text{if } \xi > 0, \\ -\mathrm{i}\hat{f}(\xi) & \text{if } \xi < 0, \end{cases}$$

where $\hat{f}$ denotes the Fourier transform of f.

(a) Why is $Hf \in L^2(\mathbb{R})$ for any $f \in \mathscr{S}(\mathbb{R})$?

(b) If $f \in \mathscr{S}(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 0.$$

[Hint: you may want to use the Riemann-Lebesgue Lemma.]

Problem 6: (10 points)

Let Δ denote the Laplace operator in $\mathbb{R}^3$.

(a) Prove that

$$\lim_{\epsilon\to 0} \int_{B_\epsilon^c} \frac{1}{|\boldsymbol{x}|} \Delta f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 4\pi f(0), \quad \forall f \in \mathscr{S}(\mathbb{R}^3)$$

where B_ϵ^c is the complement of the ball of radius ϵ centered at the origin.

(b) Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(\boldsymbol{x}), \qquad \lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = 0$$

for $f \in \mathscr{S}(\mathbb{R}^3)$.

Problem 7: (8 points)

Show that the solution to the system

$$\dot{x} = 1 + x^{10}$$

goes to infinity in finite time.

Problem 8: (12 points)

Consider the nonlinear system of ODEs:

$$\begin{aligned}\dot{x} &= y - x\left((x^2+y^2)^4 - \mu\left((x^2+y^2)^2 - 1\right) - 1\right)\\ \dot{y} &= -x - y\left((x^2+y^2)^4 - \mu\left((x^2+y^2)^2 - 1\right) - 1\right)\end{aligned}$$

(a) Rewrite the system in polar coordinates.

(b) For $0 \le \mu < 1$, show that the circular region that lies within concentric circles with radius $r_{min} = 1/2$ and $r_{max} = 2$ is a trapping region. And use the Poincaré-Bendixson theorem to show that there exists a stable limit cycle.

(c) Show that a sub-critical Hopf Bifurcation occurs at $\mu = 1$.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 22, 2009

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

For $\epsilon > 0$, let η_ϵ denote the family of *standard* mollifiers on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the functions

$$u_\epsilon = \eta_\epsilon * u \text{ in } \mathbb{R}^2.$$

Prove that

$$\epsilon \|Du_\epsilon\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)},$$

where the constant C depends on the mollifying function, but not on u.

Problem 2: (10 points)

Let $B(0,1) \subset \mathbb{R}^3$ denote the unit ball $\{|x| < 1\}$. Prove that $\log|x| \in H^1(B(0,1))$.

Problem 3: (10 points)

Prove that the continuous functions of compact support are a dense subspace of $L^2(\mathbb{R}^d)$.

Problem 4: (10 points)

There are several senses in which a sequence of bounded operators $\{T_n\}$ can converge to a bounded operator T (in a Hilbert space $\mathcal{H}$). First, there is convergence in the norm, that is, $\|T_n - T\| \to 0$, as $n \to \infty$. Next, there is a weaker convergence, which happens to be called strong convergence, that requires that $T_n f \to Tf$, as $n \to \infty$, for every vector $f \in \mathcal{H}$. Finally, there is weak convergence that requires $(T_n f, g) \to (Tf, g)$ for every pair of vectors $f, g \in \mathcal{H}$.

(a) Show by examples that weak convergence does not imply strong convergence, nor does strong convergence imply convergence in norm.

(b) Show that for any bounded operator T there is a sequence $\{T_n\}$ of bounded operators of finite rank so that $T_n \to T$ strongly as $n \to \infty$.

Problem 5: (10 points)

Let $\mathcal{H}$ be a Hilbert space. Prove the following variants of the spectral theorem.

(a) If T_1 and T_2 are two linear symmetric and compact operators on $\mathcal{H}$ that commute (that is, $T_1 T_2 = T_2 T_1$), show that they can be diagonalized simultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.

(b) A linear operator on $\mathcal{H}$ is *normal* if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized.

(c) If U is unitary, and $U = \lambda I - T$, where T is compact, then U can be diagonalized.

Problem 6: (10 points)

Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Problem 7: (10 points)

Consider the equation

$$\frac{d^2 x}{d t^2} + x - \epsilon x|x| = 0$$

(a) Find the equation for the conserved energy.

(b) Find the equilibrium points and the values of ϵ for which they exist.

(c) There are two qualitatively different phase portraits, for different values of ϵ. CLEARLY sketch and label these phase portraits.

(d) Show that there exist initial conditions, for any ϵ, for which solutions are periodic.

(e) For initial data $x(0) = a$, $\dot{x}(0) = 0$, calculate the first two terms (in ϵa) of the Taylor expansion of the period of the orbit in the limit $\epsilon a \to 0$.

Problem 8: (10 points)

Consider the system

$$\begin{aligned}
\frac{dx}{dt} &= ax + y - xf(x^2 + y^2) \\
\frac{dy}{dt} &= -x + ay - yf(x^2 + y^2)
\end{aligned}$$

where a is real, f is continuous, $f(0) = 0$ and $f(z) \geq z^{1/2}$.

(a) Show that the origin is the only equilibrium point.

(b) Study the linear stability of the origin.

(c) Show that there exists a stable limit cycle if $a > 0$ (state and use the Poincaré-Bendixson theorem).

(d) Take the special case with $f(z^2) = z$ for all $z \geq 0$ with $a > 0$. Find the limit cycle explicitly by solving the system.

Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

January 4, 2008

Instructions:

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)
Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = (-1)^n x^n (1-x).$$

(a) Show that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[0,1]$.

(b) Show that $\sum_{n=0}^{\infty} |f_n|$ converges pointwise on $[0,1]$ but not uniformly.

Problem 2: (10 points)
Consider $X = \mathbb{R}^2$ equipped with the Euclidean metric,

$$e(x,y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{1/2},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} e(x,y) & \text{if } x,\ y \text{ lie on the same ray through the origin,} \\ e(x,0) + e(0,y) & \text{otherwise.} \end{cases}$$

Here, we say that x, y lie on the same ray through the origin if $x = \lambda y$ for some positive real number $\lambda > 0$.

(a) Prove that (X, d) is a metric space.

(b) Give an example of a set that is open in (X, d) but not open in (X, e).

Problem 3: (10 points)

Suppose that $\mathscr{M}$ is a (nonzero) closed linear subspace of a Hilbert space $\mathscr{H}$ and $\phi : \mathscr{M} \to \mathbb{C}$ is a bounded linear functional on $\mathscr{M}$. Prove that there is a unique extension of ϕ to a bounded linear function on $\mathscr{H}$ with the same norm.

Problem 4: (10 points)

Suppose that $A : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator on a (complex) Hilbert space $\mathscr{H}$ with spectrum $\sigma(A) \subset \mathbb{C}$ and resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$. For $\mu \in \rho(A)$, let

$$R(\mu, A) = (\mu I - A)^{-1}$$

denote the resolvent operator of A.

(a) If $\mu \in \rho(A)$ and

$$|\nu - \mu| < \frac{1}{\|R(\mu, A)\|},$$

prove that $\nu \in \rho(A)$ and

$$R(\nu, A) = \left[I - (\mu - \nu) R(\mu, A)\right]^{-1} R(\mu, A).$$

(b) If $\mu \in \rho(A)$, prove that

$$\|R(\mu, A)\| \geq \frac{1}{d\left(\mu, \sigma(A)\right)}$$

where

$$d\left(\mu, \sigma(A)\right) = \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is the distance of μ from the spectrum of A.

Problem 5: (10 points)

Let $1 \leq p < \infty$ and let $I = (-1, 1)$ denote the open interval in $\mathbb{R}$. Find the values of α as a function of p for which the function $|x|^\alpha \in W^{1,p}(I)$.

Problem 6: (10 points)

Let $\Omega = \{x \in \mathbb{R}^3 \ : \ |x| < 1\}$ denote the unit ball in $\mathbb{R}^3$. Suppose that the sequences $\{f_k\}$ in $W^{1,4}(\Omega)$ and that $\{\vec{g}_k\}$ in $W^{1,4}(\Omega; \mathbb{R}^3)$. Suppose also that there exist functions $f \in W^{1,4}(\Omega)$ and $\vec{g}$ in $W^{1,4}(\Omega; \mathbb{R}^3)$, such that we have the weak convergence

$$f_k \rightharpoonup f \text{ in } W^{1,4}(\Omega),$$
$$\vec{g}_k \rightharpoonup \vec{g} \text{ in } W^{1,4}(\Omega; \mathbb{R}^3).$$

Show that there are subsequences $\{f_{k_j}\}$ and $\{\vec{g}_{k_j}\}$ such that we have the weak convergence

$$\vec{D} f_{k_j} \cdot \operatorname{curl} \vec{g}_{k_j} \rightharpoonup \vec{D} f \cdot \operatorname{curl} \vec{g} \quad \text{in} \quad H^{-1}(\Omega).$$

Notation. Here f is a scalar function and $\vec{g} = (g_1, g_2, g_3)$ are three-dimensional vector-valued function. $\vec{D}$ denotes the three-dimensional gradient $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\operatorname{curl}\vec{g} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \times \vec{g}$

As customary, we use $H^{-1}(\Omega)$ to denote the dual space of the Hilbert space $H_0^1(\Omega)$ consisting of those functions in $H^1(\Omega)$ which vanish on the boundary (in the sense of trace). Two useful identities are that

$$\begin{aligned} \operatorname{curl}(\vec{D}f) &= 0 \quad \text{for any scalar function } f, \\ \operatorname{div}(\operatorname{curl}\vec{w}) &= 0 \quad \text{for any vector function } \vec{w}, \end{aligned}$$

where $\operatorname{div}\vec{F} = \partial_{x_1}F_1 + \partial_{x_2}F_2 + \partial_{x_3}F_3$ denotes the usual divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$.

Hint. Test $\vec{D}f_{k_j} \cdot \operatorname{curl}\vec{g}_{k_j}$ with a function $\psi \in H_0^1(\Omega)$ and use integration by parts to argue the weak convergence.

Problem 7: (14 points)

The rotating bead on a hoop is a Hamiltonian system where

$$H(\theta, \Omega) = \frac{\Omega^2}{2} - \left(\frac{g}{R}\cos(\theta) - \frac{\omega^2}{4}\cos(2\theta)\right)$$

where θ is the angle that the bead makes from the vertical measured from "straight down," Ω is the angular velocity of the bead, ω is the angular velocity of the hoop, g is the acceleration of gravity, and R is the radius of the hoop. Recall that the Hamiltonian is conserved (it is the total energy) and the dynamics are given by

$$\dot{\theta} = \frac{\partial H}{\partial \Omega}, \quad \dot{\Omega} = -\frac{\partial H}{\partial \theta}.$$

(a) Write down the dynamical system.

(b) When the hoop is not rotating, this is exactly equivalent to the classical pendulum. Non-dimensionalize this system using a natural time scale associated with the classical pendulum. This will leave you with one parameter, call it λ which we shall use to study bifurcations.

(c) Find the value of λ_c at which a bifurcation occurs.

(d) Sketch the phase portrait for λ greater than the bifurcation value and less than the bifurcation value.

(e) Find the fixed points and classify their stability.

(f) Find the frequency of oscillation about either of the two neutrally stable fixed points for $\lambda > \lambda_c$.

(g) Sketch the phase portrait for $\lambda > \lambda_c$ if we add a damping term to the equation, i.e., $\dot{\Omega} = -\nu\Omega$ with $\nu > 0$.

Problem 8: (6 points)

Estimate the period of the limit cycle in the system

$$\ddot{x} + k\left(x^2 - 4\right)\dot{x} + x = 1$$

for $k \gg 1$. There are different ways to do this. One way to start involves recognizing the Lienard transformation, i.e. first write the system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x} + k\left(\frac{x^3}{3} - 4x\right)\right] + x = 1.$$

Second, define the quantity in square brackets to be $k\,y$. Third, write down the dynamical system for $\dot{x}$ and $\dot{y}$. From here you can find an approximate expression for the limit cycle and integrate the resulting equation to estimate the period.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 23, 2008

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)
Prove that the dual space of c_o is ℓ^1, where

$$c_o = \{x = (x_n) \text{ such that } \lim x_n = 0\}.$$

Problem 2: (10 points)
Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Problem 3: (10 points)
Let $f \in L^1(\mathbb{R})$ and V_f be the closed subspace generated by the translates of f, i.e., $V_f := \{f(\cdot - y) \mid \forall y \in \mathbb{R}\}$. Suppose $\hat{f}(\xi_0) = 0$ for some ξ_0. Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(\mathbb{R})$, then $\hat{f}$ never vanishes.

Problem 4: (10 points)

(a) State the Stone-Weierstrass theorem for a compact Hausdorff space X.

(b) Prove that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 5: (10 points)

For $r > 0$, define the dilation $d_r f : \mathbb{R} \to \mathbb{R}$ of a function $f : \mathbb{R} \to \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathscr{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, d_{1/r}\phi \rangle \qquad \text{for all test functions } \phi \in \mathscr{D}(\mathbb{R}).$$

(a) Show that the dilation of a regular distribution T_f, given by

$$\langle T_f, \phi \rangle = \int f(x)\phi(x)\,\mathrm{d}x,$$

agrees with the dilation of the corresponding function f.

(b) A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ-distribution is homogeneous of degree -1.

(c) If T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n-1$.

Problem 6: (10 points)

Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \ldots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Define $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} x_n^2 - x_n^4 \right\}$$

(a) Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \to \mathbb{R}$ equal to zero.

(b) Show that the second derivative of F at $x = 0$,

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R},$$

is positive-definite, meaning that

$$F''(0)(h, h) > 0$$

for every nonzero $h \in \ell^2(\mathbb{N})$.

(c) Show that F does not attain a local minimum at $x = 0$.

Problem 7: (10 points)

Consider the dynamical system:

$$\begin{aligned}\dot{x} &= \quad y^3 - 4x,\\ \dot{y} &= \quad y^3 - y - 3x.\end{aligned}$$

Show that if a trajectory starts at any point on the line $x = y$, then it stays on it. Otherwise, show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ on any other trajectories.

Problem 8: (10 points)

Consider the system

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0,$$

where ϵ is a small parameter, and $h(x, \dot{x}) = (x^2 - 1)\dot{x}^3$. Show that a periodic orbit exists.

[Hint: Let $\langle\cdot\rangle$ be an averaging operator for a function defined over $[0, 2\pi]$, i.e., $\langle f\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$. Calculate the averaged equation, $r' = \langle h(x, \dot{x})\sin\rangle$ and identify its fixed points. You can use $\langle\cos^{2n+1}\sin^{2m+1}\rangle = 0$, $\langle\cos^{2n}\rangle = \langle\sin^{2n}\rangle = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$ to simplify your calculations.]

Winter 2007: Applied Math Preliminary Exam
Part I: Analysis

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0,1]) \to C([0,1])$ be a bounded linear operator. Suppose that $\|S(p)\| \leq 2$ for all polynomials p. Show that S is the zero operator.

Problem 2. For $p \geq 1$, let $l^p(\mathbb{N})$ be the set of sequences (x_n) such that

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

(a) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \subseteq l^q(\mathbb{N})$.

(b) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \neq l^q(\mathbb{N})$.

Problem 3. Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{x \to 0} \lim_{y \to 0} f(x,y) = \lim_{y \to 0} \lim_{x \to 0} f(x,y);$$

in particular, both limits exist. Does it follow that

$$\lim_{(x,y) \to (0,0)} f(x,y)$$

exists?

Problem 4. Let X be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < Cd(x,y)$ for all $x \neq y$. The function f is said to be a *weak contraction* if $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$, without the constant C. The contraction mapping theorem says that if f is a contraction, then it has a fixed point. Show that the theorem also holds when f is a weak contraction and X is compact.

Problem 5. Construct the Green's function for the Dirichlet boundary-value problem

$$-u'' + 4u = f, \quad u(0) = u(2) = 0.$$

Problem 6. Let U be a unitary operator on a Hilbert space. Prove that the spectrum of U lies on the unit circle.

Winter 2007: Applied Math Preliminary Exam
Part II: ODE Theory

1. Show that the system

$$\begin{aligned}\dot{x} &= -x + y^3 - y^4 \\ \dot{y} &= -2x - y + 2xy\end{aligned}$$

has no periodic solutions. What is the asymptotic behavior, as $t \to \infty$, of the trajectory starting at $(\pi, -e^2)$? (Hint: Choose a, m and n such that $V = x^m + ay^n$ is a Liapunov function.)

2. Consider the system

$$\ddot{x} = -x^2 + (r-2)\,x + r - 1,$$

where r is a parameter.

(a) Show that there is a bifurcation at $r = r_c$ for some r_c. Find the value of r_c. What kind of bifurcation is it?

(b) Classify the fixed points of this nonlinear system. Give your reasons.

Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 25, 2007

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f\left(x^n\right) \mathrm{d}x$$

exists and evaluate the limit. Does the limit always exist if f is only assumed to be integrable?

Problem 2: (10 points)

Suppose that for each $n \in \mathbb{Z}$, we are given a real number ω_n. For each $t \in \mathbb{R}$, define a linear operator $T(t)$ on 2π-periodic functions by

$$T(t)\left(\sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}nx}\right) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega_n t} f_n \mathrm{e}^{\mathrm{i}nx},$$

where $f(x) = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}nx}$ with $f_n \in \mathbb{C}$.

(a) Show that $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a unitary map.

(b) Show that $T(s)T(t) = T(s+t)$ for all $s, t \in \mathbb{R}$.

(c) Prove that if $f \in C^\infty(\mathbb{T})$, meaning that it has continuous derivatives of all orders, then $T(t)f \in C^\infty(\mathbb{T})$.

Problem 3: (10 points)

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces, with X compactly imbedded in Y, and Y continuously imbedded in Z (meaning that: $X \subset Y \subset Z$; bounded sets in $(X, \|\cdot\|_X)$ are precompact in $(Y, \|\cdot\|_Y)$; and there is a constant M such that $\|x\|_Z \le M\|x\|_Y$ for every $x \in Y$). Prove that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|x\|_Y \le \varepsilon \|x\|_X + C(\varepsilon)\|x\|_Z \qquad \text{for every } x \in X.$$

Problem 4: (10 points)

Let $\mathcal{H}$ be the weighted L^2-space

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} e^{-|x|} |f(x)|^2 \, dx < \infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) \, dx.$$

Let $T : \mathcal{H} \to \mathcal{H}$ be the translation operator

$$(Tf)(x) = f(x+1).$$

Compute the adjoint T^* and the operator norm $\|T\|$.

Problem 5: (10 points)

(a) State the Rellich Compactness Theorem for the space $W^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Recall that the Sobolev conjugate exponent is defined as $p^* = \frac{np}{n-p}$, and that there are some constraints on the set Ω.

(b) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^3$ open, bounded, and smooth. Show that there exists an $f \in H^1(\Omega)$ such that for a subsequence $\{f_{n_\ell}\}_{\ell=1}^{\infty}$,

$$f_{n_\ell} D f_{n_\ell} \rightharpoonup f Df \quad \text{weakly in } L^2(\Omega),$$

where $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$ denotes the (weak) gradient operator.

Problem 6: (10 points)

Let $\Omega := B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^2$ denote the open ball of radius $\frac{1}{2}$. For $x = (x_1, x_2) \in \Omega$, let

$$u(x_1, x_2) = x_1 x_2 \left[\log\left(|\log(|x|)| \right) - \log\log 2 \right] \text{ where } |x| = \sqrt{x_1^2 + x_2^2}.$$

(a) Show that $u \in C^1(\bar{\Omega})$.

(b) Show that $\frac{\partial^2 u}{\partial x_j^2} \in C(\bar{\Omega})$ for $j = 1,2$, but that $u \notin C^2(\bar{\Omega})$.

(c) Using the elliptic regularity theorem for the Dirichlet problem on the disc, show that $u \in H^2(\Omega)$.

Problem 7: (8 points)

Write down the general solution for the following two linear dynamical systems. Draw the phase plane with trajectories, and eigenvectors clearly labeled. Compare the two systems specifically paying attention to the nature of their eigenvectors and eigenvalues. Explicitly write the solution to both systems for the initial condition $x(0) = 0$, $y(0) = 1$. Sketch the plot of $x(t)$ on the same axes for each of these solutions.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Problem 8: (12 points)

Consider the second order dynamical system

$$\begin{aligned} \dot{x} &= x\left[x(1-x) - y\right] \\ \dot{y} &= y(x-a) \end{aligned}$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and a is a positive control parameter.

(a) Sketch the nullclines in the positive quadrant of the x, y plane.

(b) Find the fixed points and classify their (linearized) stability.

(c) Sketch the phase portrait for $a > 1$. What happens to the predators in this case?

(d) Find the value of a for which a Hopf bifurcation occurs.

(e) Estimate the frequency of the limit cycle oscillations for a near the bifurcation.

(f) Sketch all the topologically distinct phase portraits for $0 < a < 1$.

Fall 2006: PhD Applied Math Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a given continuous function. Let $T_k : C([0,1]) \to C([0,1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x,y)f(y)\,dy$.

(a) Show that T_k is a bounded operator.

(b) Find an expression for $\|T_k\|$ in terms of k.

(c) What is $\|T_k\|$ if $k(x,y) = x^2y^3$?

Problem 2. Let X be a metric space.

(a) Define X is sequentially compact.

(b) Define X is a complete metric space.

(c) Prove that a sequentially compact metric space X is complete.

(d) Let $B = \{\, x \;:\; \|x\|_2 \le 1 \,\}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that B is not sequentially compact.

Problem 3. Give an example of a Banach space X and a sequence (x_n) of elements in X such that $\sum_{n=1}^\infty x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^\infty |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n\in\mathbb{Z}}$ be the Fourier coefficient sequence of f; here, $\mathbb{T} := \{\, z \in \mathbb{C} \;:\; |z| = 1 \,\}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that f is continuous? (In other words, is there a continuous function that is equivalent to f in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions T of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^1)$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0,1])$ *symmetric, essentially self-adjoint, self-adjoint*? Justify your answers.

(a) $D_A = C^1[0,1]$
(the space of continuously differentiable complex-valued functions on $[0,1]$)

(b) $D_A = \{\, f \in C^1[0,1] \; : \; f(0) = f(1) \,\}$

(c) $D_A = \{\, f \in C^1[0,1] \; : \; f(0) = f(1) = 0 \,\}$

Problem 7. Consider the system

$$\begin{aligned} \dot{x} &= -y + axe^{x^2+y^2} \\ \dot{y} &= x + aye^{x^2+y^2} \end{aligned}$$

near the fixed point $(0,0)$, where a is a parameter.

(a) Classify the stability of the fixed point $(0,0)$ in its linearized system.

(b) Classify the stability of the origin in the original nonlinear system. (Hint: Express the system in polar coordinates, and recall that $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$.)

Problem 8. Show that the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y\left(4 - x^2 - 4y^2\right) \end{aligned}$$

has at least one closed orbit in the annulus

$$1 \leq x^2 + y^2 \leq 4.$$

GGAM PRELIMINARY EXAM
January 3, 2005

Write solutions on the paper provided, putting each problem on a separate page. Justify all of your mathematics. Print your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

1. Consider the equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{1}$$

with parameter μ.

(a) Show that equation (1) is equivalent to the system

$$\begin{aligned} \dot{x}(t) &= y - g(x) \\ \dot{y}(t) &= -x, \end{aligned} \tag{2}$$

with $g(x) = \mu(\frac{1}{3}x^3 - x)$.

(b) Consider the function $V(x,y) = \frac{1}{2}(x^2 + y^2)$. Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of system (2). Calculate the time derivative of $V(x(t), y(t))$. Describe the regions where $\frac{dV}{dt}$ is negative and positive. (These regions should depend on μ.)

(c) For which values of μ does the fixed point $(0,0)$ change its stability?

(d) For which values of μ does the system have a limit cycle? Explain your answer using the Poincare-Bendixson Theorem.

2. Consider the system

$$\dot{x} = x[x(1-x) - y] \qquad \dot{y} = y(x-2), \tag{3}$$

where $x \geq 0$ and $y \in \mathbb{R}$.

(a) What is the asymptotic behavior, as $t \to \infty$, of the trajectory starting at $(2,1)$?

(b) What is the asymptotic behavior, as $t \to \infty$, of the trajectory starting at $(2,-4)$? (Explain your answers.)

3. Suppose that (P_n) is a sequence of orthogonal projections on a Hilbert space $\mathcal{H}$ such that

$$\operatorname{ran} P_{n+1} \supset \operatorname{ran} P_n, \qquad \bigcup_{n=1}^{\infty} \operatorname{ran} P_n = \mathcal{H},$$

where ran P_n denotes the range of P_n.

(a) Prove that for every $x \in \mathcal{H}$, the sequence $(P_n x)$ converges to x as $n \to \infty$ with respect to the norm on $\mathcal{H}$.

(b) Prove that (P_n) does not converge to the identity operator I with respect to the operator norm on $\mathcal{B}(\mathcal{H})$ unless $P_n = I$ for some n.

4. If A is a subset of a metric space X, with metric $d : X \times X \to \mathbb{R}$, define the 'distance from A' function $f_A : X \to \mathbb{R}$ by

$$f_A(x) = \inf_{a \in A} d(x, a).$$

(a) Prove that f_A is continuous.

(b) Prove that if A is a closed subset of X, then

$$A = \{x \in X \mid f_A(x) = 0\}.$$

(c) A subset of a metric space is said to be a G_δ if it is a countable intersection of open sets. Prove that every closed subset of a metric space is a G_δ.

5. Suppose that $f : [0, 1] \to \mathbb{R}$ is a function with the property that there exist constants $M > 0$, $\alpha > 1$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha \qquad \text{for all } x, y \in [0, 1],$$

Prove that f is a constant.

6. (a) Briefly define a distribution on $\mathbb{R}$, and define what it means for a sequence (T_n) of distributions to converge.

(b) Show that the following sequence converges in the sense of distributions, and determine its limit:

$$T_n = \sin(nx).$$

(c) Show that the following sequence converges in the sense of distributions, and determine its limit:

$$T_n = n\left(\delta_{1/n} - \delta_{-1/n}\right).$$

Here, δ_a denotes the delta-distribution supported at a, defined by

$$\langle \delta_a, \phi \rangle = \phi(a).$$

7. (a) Define the subspace $H^1([0,1])$ of $L^2[0,1]$ using Fourier series.

(b) Give the definition of the *spectrum* of a linear operator A (not necessarily bounded) defined on a (dense linear) domain $\mathcal{D}(A)$ in a Hilbert space $\mathcal{H}$.

(c) Let A be the (unbounded) linear operator on $\mathcal{H} = L^2([0,1])$ with domain $\mathcal{D}(A) = \{u \in H^1([0,1]) \mid u(0) = u(1)\}$, and $Au = \frac{1}{2\pi i}u'$, where u' is the weak derivative of u. Prove that the spectrum of A defined above is the set of integers.

Analysis Preliminary Exam
Applied Mathematics, September, 2004

Write solutions on the paper provided, putting each problem on a separate page. Justify all of your mathematics. Print your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

(1) Consider the first order ordinary differential equation

$$\dot{x} = x^3 - 3x^2 + x, \tag{1}$$

where $x = x(t)$ is a real valued function of real variable t. Recall that a subset S of $\mathbf{R}$ is said to be a *positively invariant region* for (1) if, whenever $x(0) \in S$, then $x(t) \in S$ for all $t \geq 0$. Prove that $[0,4] = \{x : 0 \leq x \leq 4\}$ is a positively invariant region for (1).

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(2) Consider the second order ordinary differential equation

$$\ddot{x} + x^3 = 0. \tag{2}$$

Prove that (2) has no unbounded solutions.

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(3) Consider the following equation for an unknown function $f : [0,1] \to \mathbf{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)dy + \frac{1}{2} sin(f(x)). \tag{3}$$

Prove that there exists a $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all $g \in C([0,1])$, (3) has a unique continuous solution.

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(4) Let $\mathbf{X}$ be a normed linear space, and let $\mathbf{X}^*$ be its dual.

(a) State the Hahn-Banach Theorem for $\mathbf{X}$.

(b) Use the Hahn-Banach Theorem to prove that if the pair $x, y \in \mathbf{X}$ has the property that $\phi(x) = \phi(y)$ for all $\phi \in \mathbf{X}^*$, then $x = y$.

---

(5) Suppose that $f \in H^1([a,b])$ and $f(a) = f(b) = 0$. Prove the *Poincare Inequality*

$$\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 dx. \tag{4}$$

---

(6) Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$.

(a) Give the definitions of the *point*, *continuous* and *residual* spectrum of A.

(b) A complex number λ is said to belong to the *approximate spectrum* of A if there is a sequence (x_n) of vectors in $\mathcal{H}$ such that $||x_n|| = 1$, and $(A - \lambda I)x_n \to 0$ as $n \to \infty$. Prove that the appoximate spectrum is contained within the spectrum, and contains the point and continuous spectrum.

(c) Give an example to show that a point in the residual spectrum need not belong to the approximate spectrum.

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(7) Let $f_n : [0, +\infty) \to \mathbf{R}$ be a sequence of continuous, $n = 1, 2, 3...$, and suppose that there exists a constant $c \geq 0$ such that $f_n(x) \geq -c$ for all $x \geq 0$, $n \geq 1$. Prove that, for all $n \geq 1$, we have

$$\int_0^\infty e^{-f_n(x)-x} dx \geq e^{\int_0^\infty e^{-x} f_n(x) dx}. \tag{5}$$

(Hint: $d\mu = e^{-x} dx$ is a probability measure.)

Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(3 January 2003)

This exam has 2 numbered pages and is closed book. The Analysis portion of this exam is Problems 1-6. The ODE portion is Problems 7-9.

Problem 1. Let $f(x)$ be a real-valued function from $L^2(R^1)$ and

$$\alpha = \int_{-\infty}^{+\infty} f(x) \exp(-x^2) dx, \quad \beta = \int_{-\infty}^{+\infty} f(x) x \exp(-x^2) dx.$$

a) (5 points) Prove that $\alpha^2 < \pi \int_{-\infty}^{+\infty} f^2(x) dx$.

b) (5 points) Prove that $\alpha^2 + 2\beta^2 < \pi \int_{-\infty}^{+\infty} f^2(x) dx$.

Problem 2. Calculate the Fourier coefficients of the functions $f(x)$ and $g(x)$ in $L^2(0, 2\pi)$ where

a) (5 points) $f(x) = \cos^6(x)$,

b) (5 points) $g(x) = x - \pi$.

Problem 3.

a) (5 points) Prove or disprove that R^n equipped with the usual Euclidean norm is separable (i.e. it has a countable dense subset). Does the answer depend on the particular choice of norm in R^n?

b) (5 points) Prove that $l^\infty(\mathbb{Z})$ with the usual sup norm is not a separable space.

Problem 4. (10 points) Find all non-negative integers n and m such that $x^n \frac{d^m \delta(x)}{dx^m}$ is identically zero, where $\delta(x)$ is the delta function.

Problem 5. Consider the Hilbert space $L^2[-1, 1]$.

a) (5 points) Find the orthogonal complement of the space of all polynomials. Hint: Use the Stone-Weierstrass theorem.

b) (5 points) Find the orthogonal complement of the space of polynomials in x^2.

Problem 6. (15 points) Consider the space of all polynomials on $[0, 1]$ vanishing at the origin with the sup norm. Prove that the space is not complete and find its completion.

Problem 7. Consider the 2-d system

$$x' = x, \quad y' = -y + x^2.$$

a) (5 points) Show that the system has a saddle point at (0,0) and its stable manifold is the y-axis.

b) (5 points) Let (x, y) be a point on the unstable manifold and close to (0,0). Write the $y = u(x)$ and assume

$$u(x) = \sum_{k\geq 1} c_k x^k.$$

Determine the coefficients c_k (and thus $u(x)$) by substituting the expression into the equations.

c) (5 points) Check that your analytical result produces a curve with the same shape as the stable manifold shown in the figure.

Problem 8. (10 points) Show that $x' = y, y' = -x - x^3$ has a fixed point at the origin that is a center (i.e. the Jacobian has purely imaginary eigenvalues). Are the trajectories in a small neighborhood of the origin closed (i.e. periodic orbits)? Prove your answer.

Problem 9. (10 points) Sketch an argument for the existence of a periodic orbit for the system

$$x'' + x + \epsilon(x^2 - 1)x' = 0$$

with a small positive parameter ϵ.

Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(24 September 2003)

This exam has 3 numbered pages and is closed book. Please provide *complete* arguments. State or cite by name any major theorems you use. The Analysis portion of this exam is Problems 1-6. The ODE portion is Problems 7 & 8.

Problem 1. Recall the definition of the Gaussian distribution with variance $\sigma^2 > 0$:

$$p_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}), \ x \in \mathbb{R}.$$

For $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, define

$$(Tf)(x) = \sqrt{2} \int_{-\infty}^{+\infty} p_{1/2}(y) f(\sqrt{2}(x-y))\, dy.$$

a) Prove that p_1 is a fixed point of T.

b) Prove that for all $c > 0$, there is exactly one fixed point of T in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, say f, such that $\|f\|_{L^1} = c$.

c) Let $g \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ be a non-negative function. Show that the sequence $T^n g$ converges in $L^1(\mathbb{R})$ as $n \to \infty$, and find its limit.

Problem 2. Let $(t_n)_{n\geq 1}$ be a sequence of non-negative real numbers such that $\sum_{n\geq 1} t_n^{3/2} = 1$. Let (a_n) be a sequence of complex numbers satisfying

$$\sum_{n\geq 1} |a_n|^3 < +\infty. \tag{1}$$

Define $f_n \in C([0,1])$, by

$$f_n(x) = \sum_{m=1}^{n} t_m a_m \sin(m\pi x)$$

Prove that the set

$$A = \{f_n \mid n \geq 1\}$$

is precompact in $C([0,1])$ with the supremum norm.

Problem 3.

a) Let X^{-1} be the distributional limit, as $\epsilon \to 0$, of the sequence of functions

$$F_\epsilon(x) = \begin{cases} \frac{1}{x}, & |x| > \epsilon \\ 0, & |x| < \epsilon \end{cases}$$

Show that X^{-1} is the distributional derivative of the function $f(x) = \log|x|$.

b) Show that the distributional limit, as $\epsilon \to 0$, of the following sequence

$$f_\epsilon(x) = \frac{1}{x - i\epsilon}, \qquad \epsilon > 0$$

is $X^{-1} + \pi i \delta$.

Problem 4. Let $h > 0$, and consider the following differential-difference initial-value problem, where $u(x,t)$ and $f(x)$ are 2π-periodic functions of x:

$$\begin{aligned} u_t(x,t) &= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}, \\ u(x,0) &= f(x). \end{aligned}$$

a) (10 points) Use Fourier series to solve for $u(x,t)$ when $f(x)$ is square-integrable.

b) (5 points) How does the smoothness of $u(\cdot,t)$ for $t > 0$ compare with the smoothness of $f(\cdot)$?

c) (5 points) Discuss briefly what happens to your solution in the limit $h \to 0$.

Problem 5.

a) (5 points) Define "orthogonal projection on a Hilbert space".

b) (10 points) Suppose that P and Q are orthogonal projections with ranges $\mathcal{M}$ and $\mathcal{N}$, respectively. If $PQ = QP$, prove that $R = P + Q - PQ$ is an orthogonal projection. What is its range?

Problem 6.

a) (5 points) Define strong and weak convergence in a Hilbert space.

b) (5 points) Suppose that $(x_n)_{n=1}^\infty$ is an orthogonal sequence in a Hilbert space, meaning that x_n is orthogonal to x_m for $n \neq m$. Prove that the following statements are equivalent:

(i) $\displaystyle\sum_{n=1}^\infty x_n$ converges strongly;

(ii) $\displaystyle\sum_{n=1}^\infty x_n$ converges weakly;

$$\text{(iii)} \qquad \sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

c) (5 points) Give an example to show that if the sequence $(x_n)_{n=1}^{\infty}$ is not orthogonal, then $\sum_{n=1}^{\infty} x_n$ may converge weakly but not strongly.

Problem 7. Consider the system

$$\begin{aligned} \dot{q} &= 4p^3 - 4pq \\ \dot{p} &= 2p^2 - 3q^2 \end{aligned}$$

a) Show that the function $H(q,p) = p^4 - 2p^2q + q^3$ is a conserved quantity for this system.

b) Compute the linearization of the system at the fixed point

$$(q^*, p^*) = \left(\frac{2}{3}, \sqrt{\frac{2}{3}}\right)$$

What type of fixed point is this? Sketch the behavior of the full system in a small neighborhood of the fixed point.

Problem 8. Consider the one-dimensional system

$$\dot{x} = x + \frac{rx}{1+x^2}$$

a) Compute the location of all fixed points as a function of $r \in \mathbb{R}$.

b) Plot the phase portrait when $r = -2$.

c) Plot a bifurcation diagram for the system. At what values of x and r does the bifurcation occur? What type of bifurcation is it?

d) Describe what would happen to the system's solution if it starts at $x = 1/2$ and $r = -2$, and then r is very slowly increased? Assume that the system dynamics are much faster than the change in r.

Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(3 January 2002)

Problem 1. (5 points) Let V be a metric space with the property that every sequence $(x_k)_{k\geq 1}$ such that

$$d\left(x^k, x^l\right) < 3^{-k} \qquad \text{for } l \geq k \geq 1$$

is convergent. Prove that V is complete.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded function, and define

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) e^{-y^2/2}\, dy$$

where $-\infty < x < \infty$ and $t \geq 0$.

a) (5 points) Show that $u(x,t)$ is a solution of the initial value problem

$$\begin{aligned} u_t &= u_{xx} - xu_x \qquad & -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x) \qquad & -\infty < x < \infty. \end{aligned}$$

b) (5 points) What is the asymptotic behavior of $u(x,t)$ as $t \to \infty$?

Problem 3. Let $\mathcal{H}$ denote a Hilbert space with inner product $(\cdot, \cdot)$. For any two vectors $f, g \in \mathcal{H}$, define the operator $f \otimes g$ by

$$(f \otimes g)\, v = f\ (g, v)\,, v \in \mathcal{H}$$

Let $\{\varphi_k\}_{k=0,1,2,\ldots}$ be an orthornormal basis for $\mathcal{H}$. For each positive integer N define the operator K_N by

$$K_N = \sum_{k=0}^{N-1} \varphi_k \otimes \varphi_k$$

a) (5 points) What is the dimension of the range of K_N?

b) (5 points) Prove that K_N is a projection operator.

Problem 4. Let $\{f_n\}$ denote a sequence of vectors in the Hilbert space $\mathcal{H}$.

a) (5 points) Define the notion of *strong convergence* of this sequence to a vector $f \in \mathcal{H}$.

b) (5 points) Define the notion of *weak convergence* of this sequence to a vector $f \in \mathcal{H}$.

c) (5 points) Give an example *to show* that a sequence can converge weakly but not strongly. Be sure to show that your sequence converges weakly but does not converge strongly.

d) (5 points) Let A be a bounded operator on $\mathcal{H}$ and $\{f_n\}$ a sequence of vectors that converge strongly to f. Prove that $\{Af_n\}$ converges strongly to Af.

Problem 5. Let A denote a bounded operator on the Hilbert space $\mathcal{H}$.

a) (5 points) Define the adjoint operator A^*.

b) (5 points) What does it mean for the operator A to be self-adjoint?

c) (5 points) Prove that if λ is any eigenvalue of a self-adjoint operator A, then λ is a real number.

d) (5 points) Let A be self-adjoint and λ a complex number with nonzero imaginary part. Define the resolvent operator R_λ. What can you say about R_λ?

Problem 6. Consider a non-linear autonomous system of ODE's

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad .$$

a) (5 points) Define a *rest point* of this system, and explain in what sense it corresponds to a solution.

b) (5 points) Give the precise definition of *stable* and *asymptotically stable* rest point.

Problem 7. The orbit of a planet in general relativity follows a trajectory of the ODE

$$\ddot{u} = -P'(u) \quad ,$$

where $u = r^{-1}$, $r =$ distance from the planet to the sun (assumed to be point masses), and

$$P(u) = -Cu(u - u_2)(u - u_3) \quad .$$

Here $C > 0$ is a constant, and $0 < u_2 < u_3 < \infty$, are critical values of u.

a) (5 points) Define the energy and sketch the phase portrait, noting the character of all rest points.

b) (5 points) Show which values of the energy correspond to bounded periodic orbits.

c) (5 points) Use the phase portrait to argue that any orbit sufficiently close to the sun, with sufficiently small energy, will fall into the sun (evidence of a black hole).

Problem 8. Consider the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \quad ,$$

where

$$A = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} \quad .$$

a) (5 points) Find $\exp A$, and use it to give a formula for the solution set.

b) (5 points) Prove that

$$\lim_{t \to +\infty} \mathbf{x}(t) = 0$$

for any solution $\mathbf{x}(t)$.

Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(25 September 2002)

This exam has 3 pages (9 problems) and is closed book.

Problem 1. (10 points) Prove that R^1 with each of the metrics

(i) $\rho(x, y) = |\arctan(x) - \arctan(y)|$

(ii) $\rho(x, y) = |\exp(x) - \exp(y)|$

is incomplete and find its completion in each case.

Problem 2. (10 points) Let $\{c_k\}_{k=-\infty}^{+\infty}$ be the Fourier coefficients of an integrable function $f \in L^1(T^1)$ on a unit circle. Find the Fourier coefficients of the Steklov function

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy.$$

What can be said about their behavior as $h \to 0$?

Problem 3. (5 points) Prove or disprove that $C[0, 1]$ with the usual sup norm is a Hilbert space.

Problem 4. (10 points) Consider a sequence of functions $f_n(x) = \frac{1}{n^{10}+1} \times \exp(-nx^2)$ in the Schwartz space $S(R^1)$. Prove or disprove that it converges to zero (in the Schwartz space topology) as $n \to \infty$.

Problem 5. (10 points) Let λ be an eigenvalue of the Fourier transform on R^1.

(i) Prove that the absolute value of λ is one.

(ii) Prove that $\lambda^4 = 1$.

Problem 6. (10 points) Consider a convolution with $f(x) = \frac{sin(\pi x)}{\pi x}$ as a linear operator on $L^2(R^1)$. Prove that it is a self-adjoint operator and find its norm.

Problem 7. (10 points) Let $\{e_k\}_{k=1}^{\infty}$ be a natural orthonormal basis in $l^2(Z^1_+)$. Define a sequence of linear operators in $l^2(Z^1_+)$ by the formula

$$A_n e_k = \delta(n,k) e_1, \quad k = 1, 2, \ldots, \; n = 1, 2, \ldots,$$

where $\delta(n,k)$ is the Kronecker symbol (i.e. it is equal to one when $n = k$ and it is equal to zero otherwise). Prove that

(i) $\|A_n\| = 1$.

(ii) A_n strongly converges to zero as $n \to \infty$ (i.e. for any vector x one has $A_n x \to 0$.)

Problem 8. For each of the equations

$$(i)\; x' = rx - 4x^3, \quad (ii)\; x' = r^2 - x^2$$

answer the following questions.

a) (5 points) Determine the bifurcation point of r and sketch the different types of vector field, including the fixed point(s), for r smaller than, equal to and bigger than the bifurcation point.

b) (5 points) Sketch the bifurcation diagram (i.e. the fixed point(s) versus r) and indicate the stability of the fixed point(s) on the diagram.

Problem 9. Consider the equation

$$x'' + \mu(x^2 - 1)x' + x = 0$$

and answer the following questions.

a) (5 points) Reduce the second order equation to a system of 1-st order equations by introducing a new variable.

b) (5 points) For the equilibrium point $x = 0, x' = 0$ find a Lyapunov function (i.e. a function which is 0 at the equilibrium but otherwise strictly positive or negative in a neighborhood of the equilibrium and changes its value monotonically along any trajectory in that neighborhood).

c) (5 points) Determine the stability of the equilibrium point (your answer should depend on μ). At what value of μ does the equilibrium change its stability?

d) (5 points) What is the range of μ for which the system has a stable limit cycle?

Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(September 27, 2001)

Problem 1. (10 pts)
State the Riesz Representation Theorem for Hilbert spaces.

Problem 2. (10 pts)
Let T denote a linear operator on the Hilbert space $\mathcal{H}$. What does it mean for T to be *bounded*? Give an example of a bounded operator on $\mathcal{H} = L^2(-\infty, \infty)$ and give an example of an *unbounded* (linear) operator on the same space.

Problem 3. (10 pts)
Define a sequence of functions $p_n \in C([-1,1])$ by

$$\begin{aligned} p_{n+1}(x) &= \frac{1}{2}\left[p_n^2(x) + 1 - x^2\right] \qquad n = 0, 1, 2, \dots, \\ p_0(x) &= 1. \end{aligned}$$

Show that (p_n) is a monotone decreasing sequence of nonnegative polynomials. State Dini's theorem, and deduce that (p_n) converges uniformly to the function $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = 1 - |x|.$$

Problem 4. (5 pts)
Let X be a metric space with the property that every sequence (x_n) such that

$$d\left(x^m, x^n\right) < 2^{-m} \qquad \text{for } n \geq m$$

is convergent. Prove that X is complete.

Problem 5. (15 pts)
(a) (5 pts) Let T denote a self-adjoint and compact (sometimes called completely continuous) operator on the Hilbert space $\mathcal{H}$. (T need not be bounded.) State the spectral theorem as it applies to the operator T. Be sure to define all your symbols. (Think of this as a small essay.)
(b) (5 pts) Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Ku(x) = \int_0^1 k(x,y)u(y)\,dy, \qquad k(x,y) = \min(x,y).$$

Is this operator K self-adjoint? Is K also compact? Explain why or why not.
(c) (5 pts) Compute the eigenvalues of K defined as above. Show that K is a positive operator.

Problem 6. (15 pts)
(a) (5 pts) You may assume as given that the *oscillator wave functions*,

$$\varphi_k(x) = \frac{1}{\sqrt{2^k k!\,\sqrt{\pi}}} \exp(-x^2/2) H_k(x), \; k = 0, 1, 2, \ldots,$$

($H_k(x)$ are the Hermite polynomials) form a complete orthonormal system of the Hilbert space $L^2(-\infty, \infty)$. Explain why the identity

$$2 = \sum_{k=0}^{\infty} \left(\int_{-1}^{1} \varphi_k(x)\,dx \right)^2$$

must be satisfied. (Hint: The solution does not require the computation of difficult integrals!)
(b) (5 pts) For what values of $\varepsilon \in \mathbb{R}$ are the sequences

$$f = \left\{ \frac{1}{n^{\varepsilon}} \right\}_{n=1}^{\infty}$$

elements of the Hilbert space ℓ^2?
(c) (5 pts) For what values of $\varepsilon \in \mathbb{R}$ are the functions

$$f(x) = \begin{cases} 0 & : \; x = 0 \\ 1/x^{\varepsilon} & : \; 0 < x \le 1 \end{cases}$$

elements of the Hilbert space $L^2([0,1])$? If you change the value of f at $x = 0$, can this change your answer? Explain why.

Problem 7. (15 pts)
(a) (5 pts) Define the potential energy and total energy for the system

$$x'' = -x^3 - x^2 + x + 1 \qquad (*)$$

(b) (5 pts) Plot the potential energy, being careful with the max/min points.
(c) (5 pts) Use the graph in (b) to construct a phase portrait for solutions of $(*)$. Justify your diagram.

Problem 8. (20 pts)
Consider the system

$$\begin{aligned} x' &= -x - xy^2 \\ y' &= -x^2 y \end{aligned}$$

(a) (10 pts) Linearize the system about rest point $(0,0)$, and determine the stability of this rest point for the linearized system.
(b) (10 pts) Prove that the rest point $(0,0)$ is *stable* for this nonlinear system.

GGAM Prelim Exam 1–00

ODE–119A

Problems + Solutions

#6 (a) Prove that

$$E = \frac{1}{2}\dot{x}^2 + P(x)$$

is constant along solutions of

$$\ddot{x} = -P'(x)$$

(b) Consider the 2nd order ODE

$$\ddot{x} = -\cos x + 5x^2. \qquad (*)$$

~~Find a~~ Use the energy to find a first order ODE that describes solutions of (*). Demonstrate that solutions of this scalar ODE are indeed solutions of (*). ~~Discuss the correspondence of the initial conditions~~

(c) Discuss the correspondence of the initial conditions

(#7) Assume $\underset{\sim}{x}(t) = (x, y)$ satisfies the DE

$$\frac{dx}{dt} = \lambda x + y + y^2 \qquad (1)$$

$$\frac{dx^2}{dt} = \lambda x,$$

where $\lambda > 0$ is an arbitrary constant.

(a) Find the matrix A such that the linearization of this system at $(0,0)$ is given by

$$\frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x}. \qquad (2)$$

(b) Find a formula for the solution set of (2)

(3) graph the solution set in the xy-plane (That is, graph the phase portrait.)

#8 Prove that the point $(0,0)$ is an asymptotically stable rest point for system

$$\dot{x} = 4x^3 + 2xy^{12}$$

$$\dot{y} = 12x^2 y^{11}$$

(That is, prove that all solutions tend to $(0,0)$ as $t \to +\infty$)

PRELIMINARY EXAM
Math 119a
Fall 2000

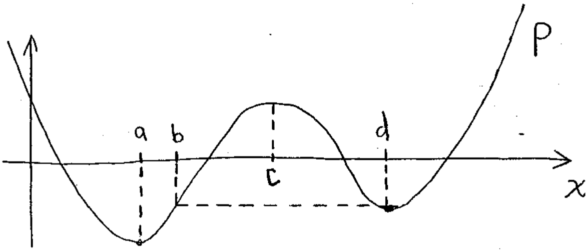
Problem 1. Find $exp(A)$ in the following cases:

(i) $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Problem 2. Consider solutions to the ODE $\ddot{x} = -P'(x)$, where $P(x)$, the potential energy, is given in graphical form below. Use the graph of P to graph the phase portrait for this ODE, (that is, graph the trajectories in the phase space, the $x\dot{x}$-plane), making sure to accurately plot the points $a - d$.



Problem 3.(Chaos and the Lorentz Equations) The Lorentz Equations are given by

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, & (1) \\ \dot{y} &= -xz + rx - y, & (2) \\ \dot{z} &= xy - bz., & (3) \end{aligned}$$

where σ, r and b are positive constants. (Lorentz used $\sigma = 10$, $b = 8/3$, $r = 28$.) Lorentz introduced this equation, (a simplified model for convection in the atmosphere), in the mid-sixties, and this is recognized as the first system of ODE's for which chaotic behavior could be rigorously demonstrated, (thereby suggesting that it is very difficult to predict the weather!)

(i) Prove that when $0 < r < 1$, there exists only one rest point $(0,0,0)$, and all solution trajectories tend to this stable rest point as $t \to \infty$. Prove this by stating the Liapunov stability theorem, and then showing that $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$ is a Liapunov function when $0 < r < 1$. (That is, $\frac{d}{dt}V(\mathbf{x}(t)) < 0$ for $(x,y,z) \neq (0,0,0)$, and thus all solution trajectories head toward the rest point $(0,0,0)$ as $t \to \infty$.) It follows that system (2) goes from *predictable* to *chaotic* as parameter values change from $r < 1$ to $r > 1$.

(ii) One of the main forces that drives the chaotic behavior of the Lorentz equations is that individual solution trajectories diverge from one another at an exponential rate, but the volume of any region is exponentially squashed, under the dynamics. But this wouldn't be such an interesting set of constraints if solution trajectories were unbounded, since then they could just go off to infinity. Thus, a main step in the analysis of (2) is to prove that solutions remain bounded for all time. Prove this by showing that solutions of (2) starting inside the ellipsoid $E = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C$, must remain inside of E so long as C is large enough so that E contains the ellipsoid $\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} \leq 1$. (Hint: Show that E defines a Liapunov function that decreases on solutions.)

Problem 1. Compute the Green's function for the BVP

$$\begin{aligned} &u'' + u = f \qquad 0 < x < 1, \\ &u(0) = a, \quad u'(1) = b, \end{aligned}$$

where $f : [0,1] \to \mathbb{R}$ is a given continuous function and a, b are given real constants. Write out the Green's function representation of the solution u.

Problem 2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given bounded, continuous functions and $\lambda \in \mathbb{R}$. Prove that if $|\lambda| < 1/2$, then there is a unique bounded, continuous solution $u : \mathbb{R} \to \mathbb{R}$ of the nonlinear integral equation

$$u(x) - \lambda \int_{-\infty}^{\infty} e^{-|x-y|} \sin\left[u(y) - g(y)\right] dy = f(x).$$

Problem 3. Let (f_n), (g_n), (h_n) be the sequences of functions in $L^2(\mathbb{R})$ defined by:

$$\begin{aligned} f_n(x) &= \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise;} \end{cases} \\ g_n(x) &= \begin{cases} 1/n & \text{if } 0 < x < n, \\ 0 & \text{otherwise;} \end{cases} \\ h_n(x) &= \begin{cases} 1 & \text{if } n < x < n+1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In each case, determine (with proof) whether or not the sequence converges: (a) strongly in $L^2(\mathbb{R})$; (b) weakly in $L^2(\mathbb{R})$; (c) in the sense of distributions.

Problem 4. Let $A : \mathcal{D}(A) \subset L^2([0,1]) \to L^2([0,1])$ be the differential operator

$$\begin{aligned} Au &= (a - c)\left(-u'' + u\right) + a'' u, \\ \mathcal{D}(A) &= \left\{u : [0,1] \to \mathbb{C} \mid u \in H^2([0,1]),\ u(0) = u(1) = 0\right\}, \end{aligned}$$

where $a : [0,1] \to \mathbb{R}$ is a given real-valued, twice continuously-differentiable function, and $c \in \mathbb{C} \setminus \mathbb{R}$ is a complex constant with nonzero imaginary part.

(a) Compute the adjoint A^* of A.

(b) If $Au = 0$, show that

$$\int_0^1 \left(|u'|^2 + |u|^2 + \frac{a''}{a-c} |u|^2 \right) dx = 0.$$

Deduce that if a'' does not change sign in the interval $[0,1]$, meaning that a has no inflection points, then the kernel of A is $\{0\}$.

Problem 5. Let $\Omega = \{(r,\theta) \mid r < 1\}$ be the unit disc in the plane, where (r,θ) are polar coordinates. The boundary of Ω is the unit circle $\mathbb{T}$. Let $\mathcal{H} \subset L^2(\mathbb{T})$ be the Hilbert space

$$\mathcal{H} = \left\{ f \in L^2(\mathbb{T}) \;\Big|\; \int_{\mathbb{T}} f(\theta)\, d\theta = 0 \right\}.$$

We define a map $N : \mathcal{H} \to \mathcal{H}$ in the following way: for $f \in L^2(\mathbb{T})$, let $u(r,\theta)$ be a solution of Laplace's equation in Ω,

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0 \qquad r < 1,$$

such that $u_\theta(1,\theta) = f(\theta)$. Then $Nf = g$ where $g(\theta) = u_r(1,\theta)$. Thus, N maps the θ-derivative of the Dirichlet data for u to the Neumann data for u. Prove that N is a well-defined, unitary map on $\mathcal{H}$.

Problem 6. The Wigner distribution $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of a Schwartz function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$W(x,k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi\left(x - \frac{y}{2}\right) \overline{\varphi\left(x + \frac{y}{2}\right)} e^{ik\cdot y}\, dy,$$

where $x, k \in \mathbb{R}^n$.

(a) Prove that

$$\int_{\mathbb{R}^n} W(x,k)\, dk = |\varphi(x)|^2 .$$

(b) Prove that

$$W(x,k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}\left(k - \frac{\ell}{2}\right) \overline{\hat{\varphi}\left(k + \frac{\ell}{2}\right)} e^{-i\ell\cdot x}\, d\ell$$

where

$$\hat{\varphi}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x)\, e^{-ik\cdot x}\, dx$$

is the Fourier transform of φ.

GGAM Preliminary Examination SEPTEMBER 1999

Explain all your answers unless otherwise instructed. State precisely or indicate by name the theorems you use in your arguments.

1. (a) Prove that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon^c}\frac{1}{|\mathbf{x}|}\Delta f(\mathbf{x})d\mathbf{x}=4\pi f(0),\quad \forall f\in\mathcal{S}(R^3)$$

where B_ε^c is the complement of the ball of radius ε centered at the origin.

(b) Find the solution u of the Poisson problem

$$\Delta u=4\pi f(\mathbf{x}),\quad \lim_{|\mathbf{x}|\to\infty}u(\mathbf{x})=0$$

for $f\in\mathcal{S}(R^3)$.

2. Consider the following sequences of functions parametrized by n.

$$\begin{aligned}
f_n(x) &= e^{i(2\sqrt{n}\pi x)}, && \text{in } L^2([0,1]) && (1)\\
f_n(x) &= \sqrt{n}e^{-nx^2}, && \text{in } L^2(R) && (2)\\
f_n(\mathbf{x}) &= \sqrt{n}e^{-n|\mathbf{x}|^2}, && \text{in } L^2(R^2) && (3)\\
f_n(x) &= ne^{-nx^2}, && \text{in } L^2(R) && (4)\\
f_n(\mathbf{x}) &= ne^{-n|\mathbf{x}|^2}, && \text{in } L^2(R^2) && (5)\\
f_n(x) &= \textstyle\sum_{k=-n}^{k=n} e^{i2k\pi x}, && \text{in } L^2([0,1]) && (6)\\
f_n(x) &= e^{-(x-n^2)^2}, && \text{in } L^2(R) && (7)
\end{aligned}$$

As n tends to infinity, which sequences converge (a) almost everywhere (b) L^2-strongly (c) L^2-weakly but not strongly (d) in distribution but not L^2-weakly?

3. Consider the following operator $A_1: D(A_1)\subset L^2([0,1])\to L^2([0,1])$, acting on complex-valued functions u, defined by

$$A_1u=\frac{1}{i}\frac{du}{dx}$$

with the domain

$$D(A_1)=\left\{u\in H^1([0,1]):u(0)=u(1)\right\}.$$

(a) Is A_1 bounded? self-adjoint? compact? Explain.

(b) Compute the eigenvalues and eigenfunctions of A_1.

4. (a) Does the Fourier series

$$g(x) := \sum_{n \in Z, n \neq 0} \frac{1}{2n\pi} e^{i2n\pi x}$$

define a function $g \in L^2([0,1])$? Explain.

(b) Consider the following operator $A_2 : L^2([0,1]) \to L^2([0,1])$ defined by

$$A_2 f = \int_0^1 g(x-y) f(y) dy, \quad f \in L^2([0,1]).$$

Is A_2 bounded? self-adjoint? compact? Explain.

(c) Find the eigenvalues and eigenfunctions of A_2. (Hint: compare with A_1 in the previous problem)

5. Which of the following statements are false? You do not need to explain your answers for this problem.

(a) The dual (as Banach space) of $L^1([0,1])$ is $L^\infty([0,1])$.

(b) The dual (as Banach space) of $L^\infty([0,1])$ is $L^\infty([0,1])$.

(c) Dirac's delta function $\delta(x)$ is in $L^1(R)$.

(d) The Fourier transform is an isometry on $L^p(R)$, $1 \leq p < \infty$.

(e) The Fourier transform has a complete set of orthonormal eigenfunctions in $L^2(R)$.

(f) For all $f \in L^1(R)$, $\hat{f}(k) \to 0$ as $k \to \infty$.

(g) For all $f \in L^2(R)$, $\hat{f}(k) \to 0$ as $k \to \infty$.

(h) Suppose $f_n \geq 0$ converges point-wise decreasingly to f. Then $\lim_{n\to\infty} \int f_n(x)dx = \int f(x)dx$.

(i) Every norm-bounded sequence in $L^2([0,1])$ has a L^2-strongly convergent subsequence.

(j) Every norm-bounded sequence in $H^1([0,1])$ has a L^2-strongly convergent subsequence.

6 Assume $\underset{\sim}{x}(t) = (x_1(t), x_2(t))$ satisfies the DE

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + x_2 + x_1 x_2 \\ \frac{dx_2}{dt} &= -3x_1 + (x_2)^2 \end{aligned}$$

Find the matrix A such that the linearization of this system at $(0,0)$ is given by

$$\frac{d\underset{\sim}{x}}{dt} = Ax.$$

7 Compute $\exp(At)$ where

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

for $\lambda \neq 0$.

8 (a) Prove that

$$E = \frac{1}{2}\dot{x}^2 + P(x)$$

is constant along solutions of

$$\ddot{x} = -P'(x).$$

(b) Consider the 2nd order ODE

$$\ddot{x} = -\cos x + Lx \qquad (*)$$

Find a first order scalar ODE that describes the solutions of (*). Argue that (in general) this first order equation requires the same number of initial conditions as (*). Demonstrate that solutions of the scalar 1st order equation are indeed (in general) solutions of (*).

9 Let A be an $n \times n$ diagonalizable matrix of constants, and assume that the eigenvalues of A all have negative real part. Prove that

$$\lim_{t \to \infty} \underset{\sim}{x}(t) = 0$$

for any solution of

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}.$$

GGAM preliminary exam (Tuesday, 6 January 1998; 3 pages)

State carefully or indicate by name the theorems you use in your arguments.

Problem 1. Define three subsets of $C([0,1])$ as follows:

$$\begin{aligned} A_1 &= \{f \in C([0,1]) \mid |f(x)| \le 1, \quad \text{for all } x \in [0,1]\} \\ A_2 &= \{f \in C^1([0,1]) \mid |f'(x)| \le 2, \quad \text{for all } x \in [0,1]\} \\ A_3 &= \{f \in C([0,1]) \mid f \text{ is a polynomial function}\} \end{aligned}$$

and consider the following three intersections

$$B_1 = A_1 \cap A_2, \quad B_2 = A_1 \cap A_3, \quad B_3 = A_1 \cap A_2 \cap A_3$$

a) Is B_1 compact, precompact but not compact, or not precompact, considered as a subset of $(C([0,1]), \|\cdot\|_{\sup})$?

b) Same question for B_2.

c) Same question for B_3.

Problem 2. Define two sequences of functions, (f_n) and (g_n), in $C([0,1])$ as follows:

$$\begin{aligned} f_n(x) &= (1 + \cos 2\pi x)^{1/n}, \quad n \ge 1 \\ g_n(x) &= (1 + \frac{1}{2}\cos 2\pi x)^{1/n}, \quad n \ge 1 \end{aligned}$$

a) What are the pointwise limits, f and g, of the sequences (f_n) and (g_n) respectively?

b) For each sequence, determine whether the convergence is uniform. Give proofs!

Problem 3. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\, dy + \frac{1}{2}\sin(f(x)) \qquad (*)$$

Prove that there exists a $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all $g \in C([0,1])$, (*) has a unique continuous solution.

Problem 4. Let X be a Banach space.

a) Prove that, for all $A \in \mathcal{B}(X)$, the following series converges in $\mathcal{B}(X)$ with the standard operator norm:

$$\exp A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

b) Prove that the map $\exp : \mathcal{B}(X) \to \mathcal{B}(X)$, as defined in **a)**, is continuous.

Problem 5. Let $f \in \mathcal{S}(\mathbb{R})$, the Schwarz space of test functions. Consider the following equation for $u \in L^2(\mathbb{R}, dx)$:

$$u''(x) + \sqrt{\pi} \int_{-\infty}^{+\infty} e^{-|x-y|} u(y)\, dy = f(x)$$

Find g such that the solution u of this equation can be expressed as

$$u(x) = \int_{-\infty}^{+\infty} g(x,y) f(y)\, dy$$

(Hint: use the Fourier Transform.)

Problem 6. For any pair of real numbers α and β define the function $F_{\alpha,\beta} : \mathbb{R}^3 \to \mathbb{R}$, by

$$F_{\alpha,\beta}(x) = \|x\|^\alpha (1 + \|x\|)^\beta$$

a) Let $p \in (1, +\infty)$. For which pairs of α and β does the functional

$$\phi_{\alpha,\beta}(f) := \int F_{\alpha,\beta}(x) f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^3)$$

extend to a continuous linear functional on $L^p(\mathbb{R}^3, dx)$? Prove your answer.

b) The same question as in **a)** for the cases $p = 1$ and $p = \infty$.

Problem 7. Consider the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$$

with initial condition $x_1(0) = a$, $x_2(0) = b$. Here, a, b, c are real numbers, and

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}.$$

a) Assume $c = 0$, and $(a, b) \neq (0, 0)$. Explain how the following limits depend on a and b:

$$\lim_{t\to\infty} x_2(t),$$

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)},$$

$$\lim_{t\to\infty} \frac{1}{t}\log x_2(t).$$

b) Assume that $c \neq 0$. Find all fixed points and discuss their stability.

Problem 8. A ball of unit mass is moving without friction in a potential well $V(x) = x^3 - 3x$. Recall that the equation of motion is $x'' = -V'(x)$, or $x' = y$, $y' = -V(x)$.

a) Show that $H(x, y) = \frac{1}{2}y^2 + V(x)$ is a conserved quantity for this sistem. Describe the behavior of trajectories by sketching the phase portrait.

b) Assume that $x'(0) = 0$, $x(0) = a$, so that the ball starts at rest at location a. Determine for which a is $x(t)$ a periodic function of t.

c) Consider now the forced equation $x'' = -V'(x) - \gamma$, where $\gamma > 0$. Determine the smallest γ for which periodic motion is no longer possible.

d) Finally, consider the damped equation $x'' = -V'(x) - bx'$, where $b > 0$. Show that $H(x, y)$ is non–increasing on every trajectory. Then show that $(0, 0)$ is a stable fixed point in this case.

GGAM preliminary exam (Tuesday, 29 Sept. 1998, 9-12 am; 4 pages)

State carefully or indicate by name the theorems you use in your arguments.

Problem 1. For each of the following sets of functions, A_1–A_4, determine which are *compact*, *precompact but not compact*, and *not precompact*, when considered as a subset of the specified topological spaces. Justify your answer.

a) A_1 is the subset of $C([0,1])$ with the supremum norm consisting of all $f \in C([0,1])$ for which there exists $N \in \mathbb{N}, a_n \in \mathbb{R}$, such that $f(x) = \sum_{n=1}^{N} a_n x^n$, and $\sum_{n=1} |a_n| \leq 1$.

b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C([0,1])$, converging uniformly to $f(x) = -x \log x$, on $[0,1]$.

$$A_2 = \{f_n \mid n \geq 1\} \cup \{f\},$$

also considered as a subset of $C([0,1])$ with the supremum norm.

c) Let $\mathcal{H}$ be a separable Hilbert space and suppose $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$.

$$A_3 = \{\frac{n-1}{n+1} e_n \mid n \geq 1\},$$

considered as a subset of $\mathcal{H}$ with its norm topology.

d) The same subset of $\mathcal{H}$,

$$A_4 = A_3,$$

but with the weak topology of the Hilbert space.

Problem 2. Define the following two sequences of functions $\mathbb{R}^2 \to \mathbb{R}$: $(f_n)_{n=1}^{\infty}$ given by

$$f_n(x) = \begin{cases} n^{1/3} & \text{if } n \leq \|x\| \leq n + \dfrac{1}{n^{5/3}} \\ 0 & \text{else} \end{cases}$$

and $(g_n)_{n=1}^{\infty}$ given by

$$g_n(x) = \begin{cases} n^{-3/5} & \text{if } \sqrt{n} \leq \|x\| \leq 2\sqrt{n} \\ 0 & \text{else} \end{cases}$$

Consider these sequences with each of the topologies given below and determine whether or not $f_n \to 0$, and/or $g_n \to 0$. Explain.

a) pointwise

b) uniformly

c) in L^2 norm

d) strongly in $L^{5/3}(\mathbb{R})$

e) weakly in $L^{5/3}(\mathbb{R})$

Problem 3. Let $n \geq$ be fixed, and let $M_n(\mathbb{C})$ denote the Banach space of all $n \times n$ matrices with complex entries and let $S^1 = [0, 2\pi)$ be the unit circle. Consider the following equation for an unknown function $A : [0, 2\pi] \to M_n(\mathbb{C})$:

$$A(\phi) = \frac{1}{2}\left((A(\phi))^2 + (B(\phi))^2\right) + \frac{\lambda}{2\pi}\int_0^{2\pi} \sin^2(\phi - \theta) A(\theta)\, d\theta \qquad (*)$$

a) Prove that, for every continuous function $B : S^1 \to M_n(\mathbb{C})$, satisfying

$$\sup_{\phi \in [0,2\pi)} \|B(\phi)\| < 1 \quad ,$$

there exists a $\lambda_0 > 0$ such that for all $\lambda \in (-\lambda_0, \lambda_0)$, (*) has a unique continuous solution $A : S^1 \to M_n(\mathbb{C})$, satisfying $\|A(\phi)\| \leq 1$, for all $\phi \in [0, 2\pi)$.

b) Show that if, in addition, $B(\phi)$ is Hermitian for all $\phi \in [0, 2\pi)$, then the solution $A(\phi)$ is necessarily Hermitian.

Problem 4. Let δ_a denote the delta distribution at $a \in \mathbb{R}$.

a) Prove that following series converges in $S^*(\mathbb{R})$ with its usual topology:

$$\phi = \sum_{n \in \mathbb{Z}} \delta_n \quad .$$

b) What is the Fourier transform, $\hat{\phi}$, of ϕ, as a tempered distribution? (Hint: $\hat{\phi}$ can also quite simply be expressed in terms of delta distributions.)

Problem 5. Consider the operator L on $L^2([-\pi, \pi])$ defined by

$$Lf(y) = \int_{-\pi}^{\pi} \sin(x-y) f(x)\, dx$$

Determine and explain whether L is

a) linear

b) bounded

c) self-adjoint

d) compact

e) Hilbert-Schmidt

f) unitary

g) normal

Problem 6. Compute the norm of the operator L of the previous problem.

Problem 7. Consider the population model:

$$\frac{dx}{dt} = x(a - bx - ky), \quad \frac{dy}{dt} = y(a - by - \sigma x) \tag{1}$$

where x, y are the populations of two competing species, and a, b, k, σ are positive constants.

a) Suppose $\sigma > k > b$. Can the two species co-exist? If so, find the equilibrium populations corresponding to co-existence.

b) For the same parameters, sketch the phase portrait by drawing all the fixed points and isoclines. Determine the long term behavior of the trajectory starting at the initial point (x_0, y_0), satisfying $a = bx_0 + ky_0$ and $\sigma x_0 + by_0 > a$.

c) Suppose $k = 3, b = 2, \sigma = 1$. Sketch the phase portrait and determine the long term behavior of the trajectory starting at the initial point $(1, 1)$.

Problem 8.

a) Which of the following is a Hamiltonian system, which is a gradient system?

$$\begin{array}{ll}(i) & x' = y, \qquad y' = -x + x^3 \\ (ii) & x' = -x + y, \quad y' = x + y.\end{array}$$

b) Find the first integral (the Hamiltonian function) for the Hamiltonian system and the potential function for the gradient system.

Problem 9. Consider the equation

$$x'' + (3x^2 - \mu)x' + x = 0 \tag{2}$$

where μ is a constant.

a) Define $y = x' + g(x)$ with $g(x) = x^3 - \mu x$. Show that eq. (2) becomes

$$\begin{aligned} x'(t) &= y - g(x) \\ y'(t) &= -x. \end{aligned} \tag{3}$$

b) Use the Liapunov function $V(x, y) = x^2 + y^2$ to determine for what values of μ is the fixed point $(0, 0)$ of equation (3) stable and what values of μ is the fixed point $(0, 0)$ unstable?

c) When $(0, 0)$ is unstable, can a trajectory starting near by $(0, 0)$ go to infinity in the long run?

d) For which value of μ does the fixed point $(0, 0)$ change its stability? What bifurcation does the system undergo when μ passes through this value (Explain the special feature of this bifurcation)?

GGAM Preliminary Exam, Winter 97

1. Consider the second order equation

$$x'' = 1 - x + \epsilon x^2 - bx'$$

where $0 < \epsilon < 1/4$.
(a) Assume that there is no damping, that is, $b = 0$. Write down the equivalent first order system and show that it is a Hamiltonian system. Classify its fixed points and sketch the phase portrait.
(b) Keep the assumption that $b = 0$, and let $x(0) = 0$, $x'(0) = a$. Show that there exists $A(\epsilon)$ so that $\lim_{t\to\infty} x(t) = \infty$ if and only if $a > A(\epsilon)$. To show how fast $A(\epsilon)$ goes to ∞ as $\epsilon \to 0$, determine the power p for which $\lim_{\epsilon\to 0} \epsilon^p A(\epsilon)$ exists.
(c) Classify the fixed points if $b > 0$.

2. Consider the system

$$x' = -2x + 2(4 - x)y$$
$$y' = -y + (4 - y)x$$

Let $S = \{(x, y) : 0 \le x \le 4, 0 \le y \le 4\}$. Assume that $(x(0), y(0)) = (a, b) \in S$.
(a) Show that $(x(t), y(t)) \in S$ for all $t \ge 0$.
(b) Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. For which choices of (a, b) is $x(t)$ increasing for $t \ge 0$?
(c) Assume that (a, b) is not one of the fixed points. Compute

$$\lim_{t\to\infty} \frac{\ln |x(t) - y(t)|}{t}$$

3. Consider the system

$$x' = \alpha x + xy$$
$$y' = \alpha y - xy$$

For every $\alpha \in (-\infty, \infty)$, determine the set of fixed points and determine whether they are asymptotically stable, neutrally stable, or unstable.

4. Let $f \in C_0^\infty(-\infty, \infty)$, the space of real valued infinitely differentiable functions with compact support equipped with the inner product

$$(f, g) = \int_{-\infty}^{\infty} q(x)\ f(x) \cdot g(x)\ dx$$

where $q(x) > 0$ is C^∞. Consider the differentiable operator

$$L\ f = a(x)\ \frac{d^2\ f}{dx^2} + b(x)\ \frac{d\ f}{dx} + c(x)\ f(x)$$

where $a(x),\ b(x),\ c(x)$ are C^∞.
(i) Under what conditions is L self adjoint?
(ii) Under what conditions is L positive definite?
(iii) Under what conditions is L skew adjoint?

5. Let $f \in C_0(-\infty, \infty)$, a continuous function with compact support. Define a family of functions in C[0,1] equipped with the L_∞ norm by

$$F = \{f_\tau(t) = f(t + \tau) : \tau \in \mathbf{R}\}$$

Show that F is precompact.

6. Let X, Y be a Banach spaces and $L : X \to Y$ be a bounded and invertible operator. Let $L_n : X \to Y$ be a sequence of bounded and invertible operators which converge strongly to L, i.e., for all x, $\|(L - L_n)x\| \to 0$ as $n \to \infty$. Suppose there exists an M such that $\|L_n^{-1}\| \leq M$ for all n. Let x and x_n be the solutions of $L\,x = f$ and $L_n\,x_n = f$. Show that $x_n \to x$.

7. Let $\{e_n\}$ be an orthonormal basis for a complex Hilbert space H. Let $\{\lambda_n\}$ be a collection of complex scalars. Formerly define a linear operator

$$L\,x = \sum_1^\infty \lambda_n\ (x, e_n)\ e_n$$

a) Show that L is well defined on a dense subset of H.
b) When is L continuous?
c) Suppose L is bounded, show it is normal.
d) When is L self adjoint?
e) When does L have a finite range?
f) Prove L is compact iff $|\lambda_n| \to 0$.

GGAM PhD preliminary exam, 23 September 1997

State carefully or indicate by name the theorems you use in your arguments.

Question 1. For each of the following sets of functions, A_1–A_4, determine which are *compact*, *precompact but not compact*, and *not precompact*, when cosidered as a subset of the specified topological spaces. Justify your answer.

a) A_1 is the following subset of $C([0,1])$ with the supremum norm :

$$A_1 = \{f \in C([0,1]) \mid f(x) = \sum_{n=1}^{N} a_n x^n, N \in \mathbb{N}, a_n \in \mathbb{R}, \sum_{n=1} n\,|a_n| \leq 1\}.$$

b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C([0,1])$, converging uniformly to $f(x) = \sqrt{x}$, on $[0,1]$.

$$A_2 = \{f_n \mid n \geq 1\} \cup \{f\},$$

also considered as a subset of $C([0,1])$ with the supremum norm.

c) Let $\mathcal{H}$ be a separable Hilbert space and suppose $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$.

$$A_3 = \{e_n \mid n \geq 1\},$$

considered as a subset of $\mathcal{H}$ with its norm topology.

d) The same subset of $\mathcal{H}$,

$$A_4 = A_3,$$

but with the weak topology of the Hilbert space.

Question 2. Define the following two sequences of functions on the real line: $(f_n)_{n=1}^{\infty}$ given by

$$f_n(x) = \begin{cases} n^{1/2} & \text{if } n \leq x \leq n + 1/n \\ 0 & \text{else} \end{cases}$$

and $(g_n)_{n=1}^{\infty}$ given by

$$g_n(x) = \begin{cases} n^{-2/3} & \text{if } n \le x \le 2n \\ 0 & \text{else} \end{cases}$$

Consider these sequences with each of the topologies given below and determine whether or not $f_n \to 0$, and/or $g_n \to 0$. Explain.

a) pointwise

b) uniformly

c) in L^2 norm

d) strongly in $L^{3/2}(\mathbb{R})$

e) weakly in $L^{3/2}(\mathbb{R})$

Question 3. Define a linear operator A on the Hilbert space $\mathcal{H} = l^2(\mathbb{N})$ as follows:

$$\mathcal{D}(A) = \{(z_n)_{n=1}^{\infty} \in l^2(\mathbb{N}) \mid \text{ only finitely many of the } z_n \text{ are non-zero}\}$$

and

$$A(z_1, z_2, \ldots) = (\sqrt{2}z_2, \sqrt{3}z_3, \sqrt{4}z_4, \ldots)$$

a) Show that A is not continuous.

b) Compute A^*.

c) Give a self-adjoint extension of A^*A.

Question 4. Consider the equation

$$f = 1 + Kf + \lambda f^2 \tag{1}$$

Where $\lambda \ge 0$, $f : [0, 1] \to \mathbb{R}$ is continuous, and K is the integral operator defined by

$$Kf(x) = \int_0^1 |x - y|(1 - |x - y|) f(y)\, dy$$

Prove that there esists a constant $\lambda_0 > 0$, such that (1), with $0 \leq \lambda < \lambda_0$, has exactly one solution in the set $\{f \in C([0,1]) \mid 1 \leq f(x) \leq 2, \text{ for all } x \in [0,1]\}$.

Question 5. Let Ω denote the unit square, i.e., $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$, and consider the Laplacian with Dirichlet boundary conditions on $L^2(\Omega)$, and call this operator Δ.

a) Compute the corresponding Green's function $g : \Omega \times \Omega \to \mathbb{R}$, as a Fourier series, i.e., find g such that for all $f \in L^2(\Omega)$, the function h defined by

$$h(x) = \int_\Omega g(x,y) f(y)\, dy$$

belongs to the domain of Δ and $\Delta h = f$.

b) Find a closed expression for g. (Hint : either sum the Fourier series for g, or use the one-dimensional case.)

Question 6. Show that the differential equation

$$y'' + 3(y')^3 + y^3 = 0$$

has an asymptotically stable zero solution. (Hint: A Liàpunov function is $y^4 + 2(y')^2$.)

Question 7. Consider the system

$$x' = 2y + 1, \quad y' = -x + 1$$

a) Sketch the phase portrait of this system.

b) Compute

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t y(s)\, ds$$

c) Take as initial condition the point $x = a, y = b$. Find

$$\max_{t\in\mathbb{R}} y(t)$$

Question 8. Consider the system

$$x' = x^2 - y^3, \quad y' = x^2(x^2 - y^3)$$

a) Find the first integral and draw the phase portrait.

b) Does this system have a simple fixed point?

GGAM Ph.D. PRELIMINARY EXAM

May 20, 1996

1. Suppose that $\alpha \in \mathbf{C}$ with
$$|\alpha| = 1.$$
Define an operator A acting on complex valued functions u by
$$\begin{aligned} &Au = \frac{1}{i}\frac{du}{dx}, \\ &A : D(A) \subset L^2(0,1) \to L^2(0,1), \\ &D(A) = \left\{u \in H^1(0,1) : u(1) = \alpha u(0)\right\}. \end{aligned}$$

(a) Show that A is self-adjoint.

(b) Compute the eigenvalues of A.

2. Let Ω be a smooth bounded domain in $\mathbf{R}^d$ with boundary $\partial\Omega$. Let $f : \Omega \to \mathbf{R}$ and $g : \partial\Omega \to \mathbf{R}$ be given functions. Show that a smooth solution u of the Dirichlet problem,
$$\begin{aligned} -\Delta u &= f(x) \quad x \in \Omega, \\ u &= g(x) \quad x \in \partial\Omega, \end{aligned}$$
is unique.

3. Are the following statements true or false? Briefly justify your answer.

(a) Let $B = \{u \in \mathbf{R}^n : \|u\| \leq 1\}$ be the closed unit ball in $\mathbf{R}^n$. Then any continuous function $f : B \to \mathbf{R}$ is bounded below and attains its infimum.

(b) Let
$$B = \left\{u(x) : \int_0^1 u(x)^2\, dx \leq 1\right\}$$
be the closed unit ball in $L^2(0,1)$. Then any continuous functional $f : B \to \mathbf{R}$ is bounded below and attains its infimum.

(c) Any continuous function $f : \mathbf{R}^n \to \mathbf{R}$ which is bounded below attains its infimum.

(d) Any continuous function $f : \mathbf{R}^n \to \mathbf{R}$ such that $f(u) \to +\infty$ as $\|u\| \to \infty$ is bounded below and attains its infimum.

4. Suppose that $u \in L^1(\mathbf{R})$. Define the Fourier transform $\hat{u}(k)$ of $u(x)$. Prove that $\hat{u}$ is continuous and that $\hat{u}(k) \to 0$ as $k \to \infty$. You can assume that the space of test functions is dense in $L^1(\mathbf{R})$.

5. Consider the system

$$x' = x^2 - y - 1$$
$$y' = xy - 2y$$

(a) Find all fixed points and for each of them determine whether it is asymptotically stable, neutrally stable, or unstable.

(b) Assume that the initial conditions are $x(0) = a$ and $y(0) = b$ with $a \in (-\infty, 1]$ and $b \in (0, \infty)$. Determine $\lim_{t \to \infty} x_1(t)$.

(c) Keep the conditions in (b). Do they imply that $x_1(t)$ is a monotone function of t for $t \geq 0$?

6. A ball of unit mass is moving in a potential well given by $V(x) = x^2 - \frac{1}{4}x^4$. The equation of motion then is $x'' = -V'(x) - bx'$, where $b \geq 0$ is a constant which measures the strength of damping.

(a) Assume first there is no energy loss, i.e. $b = 0$. Sketch the phase portrait of this system. In particular, identify a pair of heteroclinic orbits.

(b) Assume now that $b > 0$. Start a trajectory *inside* the bounded set defined by the two heteroclinic orbits from (a). Describe the behavior of this trajectory as $t \to \infty$.

(c) Keep the assumption that $b > 0$. A non-constant polynomial $p(x, x')$ is claimed to be a constant along trajectories, i.e. $p(x(t), x'(t)) = p(x(0), x'(0))$ for every t and every solution $x = x(t)$ of the equation. What can you say about this claim?

Applied Mathematics
Preliminary Examination
Winter 1995

Put your answers to questions 1–3, 4–7, and 8–9 into separate piles. Make sure your name is on each pile.

1. Two stationary charged particles p_1 and p_2 are located at distance 1 from each other. A third particle p_3 is constrained to move along the line segment joining p_1 and p_2, hence its position is determined by $x \in (0,1)$, the distance from p_1. The equation of motion is determined by the inverse–square law

$$x'' = F(x),$$

where

$$F(x) = -\frac{A}{(1-x)^2} + \frac{B}{x^2}.$$

Assume that all particles are positively charged $(A, B > 0)$ so that the forces between them are repulsive.

(a) Write down the equivalent first order system.

(b) Find the equilibrium position and velocity of the particle p_3.

(c) Draw the global phase portrait.

(d) Does the linearization of the system around the equilibrium correctly predict the system's local behavior near the equilibrium? Explain your answer.

(e) What is the behavior of the trajectories as $t \to \infty$ if a small amount of friction is introduced, i.e.

$$x'' = F(x) - \epsilon x',$$

where $\epsilon > 0$ is small?

2. Consider the system

$$\begin{aligned} x' &= x(3 - 2x - y) \\ y' &= y(2 - x - y) \end{aligned}$$

with initial conditions $x(0) = a$, $y(0) = b$, where $a, b > 0$ and $(a, b) \neq (1, 1)$. Show that the limit

$$\lim_{t\to\infty} \frac{1 - y(t)}{1 - x(t)}$$

always exists and determine its possible values.

3. Find the fixed points of the following system and for each of them determine whether it is asymptotically stable, neutrally stable, or unstable.

$$\begin{aligned} x' &= xy - 1 \\ y' &= x - y^3 \end{aligned}$$

4. Define $L : D_L \subset L^2(0,1) \to L^2(0,1)$ by

$$\begin{aligned} Lu &= -\left(e^x u'\right)' + xu, \\ D_L &= \left\{u \in C^\infty[0,1] : u'(0) = u'(1) = 0.\right\}. \end{aligned}$$

(a) Prove that L is formally self-adjoint.

(b) Define an extension $\bar{L}$ of L which is rigorously self-adjoint. You should justify your answer briefly, but a detailed proof is not required.

(c) If λ_1 is the smallest eigenvalue of L, prove that

$$0 < \lambda_1 \leq \frac{1}{2}.$$

5. Two-dimensional Minkowski space $\mathbf{M}$ is the vector space $\mathbf{R}^2$ with an inner product $\langle \cdot, \cdot \rangle$ defined by

$$\begin{aligned} &\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + x_1 y_1, \\ &\mathbf{x} = (x_0, x_1), \qquad \mathbf{y} = (y_0, y_1). \end{aligned}$$

Is this space a Hilbert space? Why? Let V be a one-dimensional subspace of $\mathbf{M}$ spanned by the vector $\mathbf{e} = (\cos\theta, \sin\theta)$, where $0 \leq \theta \leq \pi/2$. Determine the orthogonal complement,

$$V^\perp = \{\mathbf{x} \in \mathbf{M} : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \text{for all } \mathbf{y} \in \mathbf{M}\},$$

and draw a picture. Is it always true that $\mathbf{M} = V \oplus V^\perp$?

6. Define the integral operator $K : C^\infty[0, +\infty) \to C^\infty[0, +\infty)$ by

$$Ku(x) = \int_0^x \frac{u(y)}{(x-y)^{1/2}}\, dy.$$

(a) Show that

$$K^2 u = \pi \int_0^x u(y)\, dy.$$

Hint: Exchange the order of integration; you can assume that

$$\int_0^1 \frac{dt}{(1-t)^{1/2} t^{1/2}} = \pi.$$

(b) Use the result of (a) to deduce that the solution of the integral equation

$$\int_0^x \frac{u(y)}{(x-y)^{1/2}}\, dy = f(x)$$

is given by

$$u(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(y)}{(x-y)^{1/2}}\, dy.$$

7. Consider spatially periodic solutions $u(x,t)$ of the fourth order diffusion equation,

$$\begin{aligned} u_t &= \nu u_{xxxx}, \\ u(x,0) &= f(x) \\ u(x+2\pi, t) &= u(x,t). \end{aligned}$$

Here ν is a nonzero constant and $x \in \mathbf{T}$ where $\mathbf{T}$ is the unit circle. Use Fourier series to find the solution of this problem. You can assume $f \in L^2(\mathbf{T})$. Discuss the existence and smoothness of the solution for $t > 0$. Consider both $\nu > 0$ and $\nu < 0$.

8. Find a leading order approximation as $\epsilon \to 0+$ of the solution $y(x;\epsilon)$ of the boundary value problem

$$\begin{aligned} \epsilon y'' - y' + 2xy &= 0, \\ y(0;\epsilon) &= 1, \\ y(1;\epsilon) &= 0. \end{aligned}$$

9. Consider a simple harmonic oscillator whose frequency ω varies slowly in time,

$$\ddot{y} + \omega^2(\epsilon t)y = 0.$$

Use the WKB method to obtain a leading order asymptotic solution as $\epsilon \to 0+$ which is valid for $t = O(1/\epsilon)$. Let $E(\epsilon t)$ be the energy of the oscillator and define the action by $S = E/\omega$. Show that S is asymptotically constant in time.