Solution of a Certain Nonlocal ODE by Reduction to a Riemann-Hilbert Problem

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Abstract

We solve a certain integro-differential equation by reduction to a Riemann-Hilbert (RH) problem. In the process we illustrate the little-known maxim "research problems are hard". To reduce dryness we have removed all the silicon packets from this document and included an excessive number of footnotes[1].

The *Basílica de la Sagrada Família* in Barcelona was designed by the architect Antoni Gaudí. The basilica is chaotic; yet it accomplishes its goal. One feels that there is always more to see, more crevices demanding attention. The present problem may evoke similar emotions, but throughout this work, remember the Basilica; there is beauty in the chaos; the microscopic may be appalling, the macroscopic remains undeniably effective. Photo Credit: Traveldigg.com.

1 Introduction

Consider the following autonomous or non-autonomous[2] ODE

$$\begin{aligned} \frac{d}{dt}u &= F(u,t) \\ u(0) &= u_0 \in \boldsymbol{R}^d \end{aligned} \tag{1.1}$$

where $u : \boldsymbol{R} \to \boldsymbol{R}^d$ is some unknown function and $F : X \times [0, \infty) \to X$ maps some appropriatly chosen Banach space X into itself[3]; parameterized by the independent variable in the case that (1.1) is non-autonomous.

In standard ODE theory the operator $F : \boldsymbol{R}^d \to \boldsymbol{R}^d$ satisfies the relation that $F(u)(t_*) = F(u(t_*))$ at every point $t_* \in \boldsymbol{R}$ on our domain; i.e. F commutes with pointwise evaluation.

[1] All good papers have their most important content contained in the footnotes.

[2] A useful mnemonic, due to J. Nichols, is to think of the independent variable t as the "driver" of your ODE. If you cannot see the driver, then she is likely asleep (running a numerical integrator is tough work) and one hopes that the ODE is **autonomous**, so as to avoid any mathematical accidents (like dividing by zero).

[3] The author would like to take a moment to show off to his classmates and professor that he knows about fancy things like Banach spaces. And don't worry, I managed to smuggle Sobolev spaces into this project as well. This footnote is written with some very specific people in mind. You know who you are.

When solving an ODE of this type we can solve the ODE independently at any $t_* \in \boldsymbol{R}$, without needing the value of u at any other point. In fact, when undergraduates solve ODEs, they are fundamentally disentangling some rule at a **single point** and then extending their subsequent result to the entire domain.

However, the relation $F(u)(t_*) = F(u(t_*))$ is artificial; there is no *a priori* reason to expect such a relationship to hold when deriving differential equations from physical principles. Indeed, operators which do not satisfy the relation appear quite frequently in physics.

The Hilbert transform on $\boldsymbol{R}$ is an example of an operator which **does not** commute with pointwise evaluation. The Hilbert transform is the operator associated with the Fourier multiplier $-i \operatorname{sgn} \xi$, which has the singular integral representation in physical space

$$\boldsymbol{H}[u](x) = \frac{1}{\pi} \int_{\boldsymbol{R}} \frac{u(y)}{x-y} \, dy \tag{1.2}$$

taken in the principle value sense. The choice of normalization and sign are somewhat arbitrary[4]; we have chosen $\frac{1}{\pi}$ so that $\boldsymbol{H}$ is an isometry on L^2 (See for example [5]).

The Hilbert transform is an example of a **nonlocal operator**. Definitions vary, but for the sake of this project we can safely take a nonlocal operator to be an operator which does not commute with pointwise evaluation, that is $F(u)(t_*) \neq F(u(t_*))$ for some $t_* \in \boldsymbol{R}$. This is true for the Hilbert transform, since $\boldsymbol{H} \sin x = \cos x$ implies that $1 = (\boldsymbol{H} \sin)(0) \neq \boldsymbol{H}(\sin(0)) = 0$ (and we must be careful even taking the Hilbert transform of constants). In fact, the Hilbert transform illustrates an important heuristic for nonlocal operators; **knowledge of u at a point t_* is insufficient to evaluate $F(u)$ at t_*.**

Before continuing, we emphasize the following point

> standard ODE techniques rely **fundamentally** on the fact that the ODE is **local**, i.e. $F(u)(t_*) = F(u(t_*))$.

which means that our standard toolbox for solving ODEs does not apply to nonlocal ODEs.

How do these operators arise in practice? To the uninitiated they can seem like perversions of the Harmonic analyst, however they *do* actually arise frequently in physical applications. For example the Hilbert transform proves indispensable in signal processing, and has been important for understanding certain phenomena in fluid mechanics [2][5]. In hydrodynamics the Euler equations govern the evolution in time of an incompressible, inviscid fluid. They are best understood as equations of vorticity, which is defined by $\omega := \nabla \cdot u$. In this case the velocity is recovered from the vorticity by the highly nonlocal perpendicular Riesz transform $u = \nabla^{\perp}(-\Delta)^{-1}\omega$.

Understanding the Euler equations, and especially the conditions under which singularities may form, is a major unsolved problem, significantly complicated by the presence of nonlocal operators. More generally, evolution equations governed by nonlocality are poorly understood[6] and exactly solving any nonlocal ODE or PDE can help us gain insight into the behavior of solutions to such problems. In this project we will solve a nonlocal ODE exactly.

[4]The sign doesn't matter. The reader can check that the Hilbert transform is it's own skew inverse, that is $\boldsymbol{H}\boldsymbol{H} = -I$. If we have defined the Hilbert transform with some sign convention, say $\boldsymbol{H}$, then we have that $(-\boldsymbol{H})(-\boldsymbol{H}) = I$.

[5] Both of the authors of the cited paper are UC Davis Professors and incidentally they have also both been authors of letters of recommendation for this author. The author is therefore obligated to reference their work whenever possible. And thus the academic wheel keeps itself rolling. **Authors author author's offer**.

[6]For example, in hyperbolic PDE, nonlocality serves to destroy the finite speed of propagation characteristic of such problems (the preceding intended pun is intended for a specific person with whom I am currently studying a nonlocal, previously hyperbolic, PDE). In this case we must consider analagous but weaker conditions than compact support when analyzing these equations.

1.1 The Model Studied

We will focus on the following nonlocal, variable coefficient, linear ODE

$$\begin{aligned} xf' &= \boldsymbol{H}[f'''] \\ f(0) &= 0 \\ \lim_{x\to\infty} f(x) &= 1 \end{aligned} \tag{1.3}$$

which was derived by Antipov and Gao as a (simplified) model of diffusion along a grain boundary [1] (We refer the reader to Section 2 of their paper for a derivation of this model). The symmetries of this problem imply that f''' is odd, and hence we may re-write the Hilbert transform term as

$$\boldsymbol{H}[f'''](x) = 2\int_0^\infty \frac{f'''(t)}{t-x}\,dt$$

which is the first step in our analysis. Note that $\boldsymbol{H}[f''']$ is now expressed as something which resembles a Mellin transform; we will abuse this fact in the coming sections.

1.2 Some Notes

Lest the reader be led to believe that the present exposition is hollistic, it should be known that what follows is not rigorous. We try to indicate where our arguments are not fully fleshed out, but there are large swaths of the coming analysis which we blatantly ignore. In particular there is a great deal of care with which the function spaces we will be working with must be examined, our solution for the present work is to pretend that we live in a beautiful green meadow with a babbling brook; which is to say that we ignore the functional analytic framework all together. The paper by Antipov and Gao [1] carries out the analysis in full with all the gory details, and we refer the reader to them whenever we decide that "today is not the day for tracking asymptotic Hölder bounds"[7].

We also note that reducing ODEs and PDEs to Riemann Hilbert problems is a well established practice, since changing to a Riemann Hilbert problem has the effect of linearizing the original system. This is closely related to the idea of Lax-Pairs, where a nonlinear problem is converted to an overdetermined system of linear problems. The expository article by Its [4] is a good introduction to this style of reductionist thinking[8].

2 ~~Everyone's Favorite Tra~~ ~~Everyone's Second Favorite Transf~~ The Mellin Transform

You may remember the Mellin transform as the part of complex analysis which makes you long for a hot Summer day, chowing down on a juicy fruit-like object while in the park. However the transformation **is** useful for more than just puns. Recall that the Mellin transform of a function (in an appropriate function space) is given by

$$\mathcal{M}[\varphi](s) := \int_0^\infty x^{s-1}\varphi(x)\,dx \tag{2.1}$$

[7] Few days are.

[8] As we know, all of science is reducible to math, all math reducible to linear algebra, and all linear algebra reduced to solving eigenvalue problems. Humble eigenvalue, may you show us the way through this nonlinear world. Humble eigenvalue, may you give me the strength to invert this 6×6 matrix by hand.

and, in analogy to the Fourier transform, possesses the following inversion formula

$$\varphi(x) = \mathcal{M}^{-1}[\mathcal{M}[\varphi]](x) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathcal{M}[\varphi](s)\,ds \tag{2.2}$$

where c is an arbitrary real number in the range where $\mathcal{M}[\varphi]$ is analytic.

In what follows, we will find it valuable to compute the following Mellin transform

$$\mathcal{M}\left[\frac{1}{1-t^2}\right](s) = \int_0^\infty \frac{t^{s-1}}{1-t^2}\,dt = \frac{\pi}{2}\cot\left(\frac{\pi}{2}s\right)$$

which is done by first looking up the following Mellin transform in [3][9]

$$\mathcal{M}\left[\frac{1}{1-t}\right](s) = \pi\cot(\pi s)$$

analytic in the strip $0 < \Re(s) < 1$, then noting that

$$\begin{aligned}\mathcal{M}\left[\frac{1}{1-t^2}\right](s) &= \int_0^\infty \frac{t^{s-1}}{1-t^2}\,dt = \frac{1}{2}\int_0^\infty \frac{w^{\frac{s}{2}-1}}{1-w}\,dt = \frac{1}{2}\mathcal{M}\left[\frac{1}{1-t}\right]\left(\frac{s}{2}-1\right)\\ &= \frac{\pi}{2}\cot\left(\frac{\pi}{2}s\right)\end{aligned} \tag{2.3}$$

which is analytic in the strip $0 < \Re(s) < 2$.

Recall that the operation of convolution

$$(\varphi * \psi)(s) := \int_{\boldsymbol{R}} \varphi(t)\psi(s-t)\,dt$$

is dual under the Fourier transform to multiplication in the frequency space, that is

$$\mathcal{F}(\varphi * \psi) = \mathcal{F}\varphi \cdot \mathcal{F}\psi$$

We ask if there is an operation $*_{\mathcal{M}}$ which is dual under the Mellin transform to multiplication. Indeed! Such an operation does exist, and is defined by

$$(\varphi *_{\mathcal{M}} \psi)(s) := \int_0^\infty \frac{\varphi(t)\psi\left(^{s}/_{t}\right)}{t}\,dt$$

We call this operation **Mellin convolution**, and we can now prove (and by "we" I mean Titchmarsh in [6]) an analogous convolution theorem, namely

Theorem 2.1. ***(Mellin Convolution Theorem).*** *Let $\mathcal{M}$ denote the Mellin transform. Given two functions φ, ψ for which the Mellin transforms $\mathcal{M}\varphi, \mathcal{M}\psi$ are well-defined, we have the following relationship*

$$\mathcal{M}\left[(\varphi *_{\mathcal{M}} \psi)\right] = \mathcal{M}\varphi \cdot \mathcal{M}\psi \tag{2.4}$$

[9] Sorry Andrew, I found some errors in our method, which (a) explained the discrepancy we saw, (b) made the method not work, (c) didn't allow me enough time to fix.

3 Method of Reduction and Solution

We can apply our newfound knowledge of the Mellin transform to start converting our problem (1.3) into a more suitable form. We take the Mellin transform of f''' and then integrate by parts repeatedly (assuming that we can ignore boundary terms)

$$\begin{aligned} F(s) := \mathcal{M}[f'''](s) = \int_0^\infty f'''(t)t^{s-1}\,dt &= -(s-1)\int_0^\infty f''(t)t^{s-2}\,dt \\ &= (s-1)(s-2)\int_0^\infty f'(t)t^{s-3}\,dt \\ &= (s-1)(s-2)\int_0^\infty (tf'(t))t^{s-4}\,dt = \mathcal{M}[tf'](s-3) \end{aligned}$$

We now apply the inverse Mellin transform to the r.h.s. and find

$$\begin{aligned} \mathcal{M}^{-1}[\mathcal{M}[tf'](s-3)](t) &= \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} t^{-s}\mathcal{M}[tf'](s-3)\,ds \\ &= \frac{1}{2\pi}\int_{c-3-i\infty}^{c-3+i\infty} t^{-s-3}\mathcal{M}[tf'](s)\,ds = t^{-3}(tf'(t)) \end{aligned}$$

from which it follows that

$$tf'(t) = \int_{c-i\infty}^{c+i\infty} \frac{F(s)}{(s-1)(s-2)} t^{3-s}\,ds \tag{3.1}$$

Note that we now require analyiticity on the strip $c-3 < \Re(s) < c$ so that the Mellin inversion above remains valid.

Applying the Mellin convolution theorem (2.4) along with the computations (2.3) and (3.1) to our nonlocal ODE (1.3) gives

$$\boldsymbol{H}[f'''](x) = 2\int_0^\infty \frac{1}{t}\frac{f'''(t)}{1-(x/t)^2}\,dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{\pi}{2}\cot\left(\frac{\pi}{2}s\right)u^{-s}\,ds \tag{3.2}$$

It proves valuable to work at the level of the new function

$$\Phi(s) := \pi F(s)\cot\left(\frac{\pi}{2}s\right) \tag{3.3}$$

rather than directly at the level of f or F. We demand the following conditions be satisfied by the solution which we are seeking

1. The function Φ can be analytically continued in the strip

$$S := \{-3+c < \Re(s) < c\}$$

2. There is a constant M so that for any $x \in [-3+c, c]$ we have

$$\|\Phi\|_{L^\infty_x L^2(x+i\boldsymbol{R})} = \sup_{c-3\leqslant x\leqslant c}\int_{\boldsymbol{R}} |\Phi(x+it)|^2\,dt \leqslant M \tag{3.4}$$

We will operate under the assumption that these are *a priori* true, but to be rigorous we must make some sort of bootstrapping or *a posteriori* argument. This is, of course, done carefully in [1].

The first condition is imposed so that we are allowed to apply the Mellin inversion formula to recover f''' from F (see note above). The second condition says that Φ is uniformly L^2 on the strip of analyiticity [10], and is imposed as a technical consequence of the asymptotic Hölder bounds required of the solution to (1.3); again, see [1]. We introduce these two conditions mainly to illustrate to the reader that we are presenting a merely cursory analysis of the problem.

Substituting (3.1) and (3.2) into (1.3), and using Φ in lieu of F, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\Phi(s-3)-\frac{\Phi(s)}{\pi(s-1)(s-2)\cot\left(\frac{\pi}{2}s\right)}\right]t^{3-s}\,ds=0,\qquad 0<t<\infty$$

which allows us to reformulate our original ODE (1.3) as the following boundary value problem: find a function Φ which is analytic in the strip S, satisfying the constraint (3.4) along with the condition

$$\Phi(s)=G(s)\Phi(s-3),\qquad \Re(s)=c \tag{3.5}$$

where

$$G(s)=\pi(s-1)(s-2)\cot\left(\frac{\pi}{2}s\right)$$

and Φ as defined above in (3.3).

Final Steps Towards the RH Problem The astute reader is perhaps starting to see the analogy with a RH problem. We are not there yet, but will be shortly. We begin by considering where we are and where we are going. A Riemann-Hilbert problem asks us to find an analytic function which satisfies a **jump** condition over some specified line. What we have is a relationship of a function on two different lines. Imagine, however, that we take the strip S and glue the two lines $\Re(s)=c-3$ and $\Re(s)=c$ together. Then we now have a Riemann-Hilbert problem on the surface of an infinite cylinder. This is (in greatly simplified essence) what we are about to do. We will map the strip S into the entire complex plane $\boldsymbol{C}$, mapping the lines $\Re(s)=c-3$ and $\Re(s)=c$ into the same (open) cut through the complex plane.

We begin by considering the "gluing" function

$$w(s)=i\tan\left(\pi\left(\frac{1}{4}+\frac{1}{3}(s-c)\right)\right) \tag{3.6}$$

[10] This condition warrants investigation (indeed, it justified a full **remark** in the Antipov paper. This footnote takes more of a harmonic analysists' viewpoint than theirs). The original equation (1.3) obviously cannot have solutions in a nice space, say L^2. However, both $\boldsymbol{H}$ and $\mathcal{M}$ are formally zero-order on L^2 (and both isommetries with the correct normalization). Thus we expect that the L^2 regularity of f''' to be the same as F, assuming that $f'''\in L^2$ at all. The requirement that $\Phi\in L^2$ amounts to $F\in L^2_\Phi$ where L^2_Φ is the appropriately weighted L^2 space. By the isommetry we then expect that $f\in \dot{H}^3_W$, i.e. the homogeneous, order three, weighted Sobolev space. $\dot{H}^3$ is already a bit of a "pathological" space, but it is far more amenable to harmonic analysis than the problem posed in the original variables (since the solution is obviously not L^2, for example). It would be interesting to explore this further using techniques from harmonic analysis to see if one could obtain a solution in a more direct manner than the method outlined here.

which maps our strip S into the complex plane with a cut along the upper half-circle. Where $w(c - i\infty) = 1$ and $w(c + i\infty) = -1$. This transformation is not-straightforward to visualize (at least it wasn't for me), and Figure 3 should help the reader to "see" this mapping. We will use $w \in \boldsymbol{C}$ to refer to the complex numbers in the image, and s to refer to the complex numbers in the pre-image.

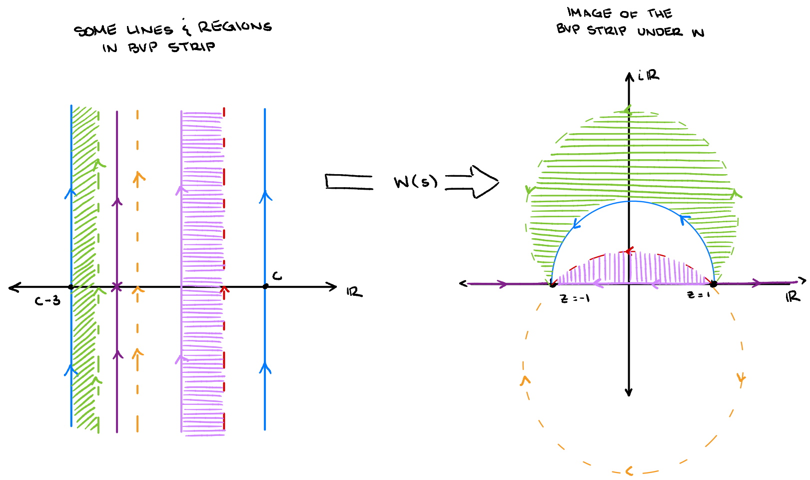

Figure 2: Graphical representation of the image of $c - 3 < \Re(s) < c$ under the transformation (3.6). Note that the point at the purple 'x' gets mapped to both $+\infty$ and $-\infty$ (think Riemann sphere, i.e. $\boldsymbol{R}P^2$).

Next we introduce the function

$$\varphi(w) = \frac{i^{-1/2}\Phi(L(w))}{(1+w)^{1/2}(1-w)^{1/2}} \tag{3.7}$$

where

$$s = L(w) = c + \frac{3i}{2\pi}\log\left(i\frac{1-w}{1+w}\right) \tag{3.8}$$

is the inverse of w. The cut is given by $\Gamma := \left\{\, |w| = 1 \,\middle|\, \Im w > 0 \,\right\}$.

We are now solving the following problem: Find a function φ, analytic on $\boldsymbol{C} \setminus \Gamma$, such that $\varphi^+(\gamma) = G(L(\gamma))\varphi^-(\gamma)$ for all $\gamma \in \Gamma$ and G is given by (3.9).

$$G(\gamma) = -\pi(L(\gamma) - 1)(L(\gamma) - 2)\cot\left(\frac{\pi}{2}L(\gamma)\right) \tag{3.9a}$$

$$L(\gamma) = c + \frac{3i}{2\pi}\log\left(i\frac{1-\gamma}{1+\gamma}\right) \tag{3.9b}$$

which satisfies the further conditions on Φ given above.

We fix the branches of the logarathmic functions $\log(\eta - 1)$ and $\log(\eta + 1)$ by restricting the arguments

$$0 < \arg(\eta + 1), \arg(\eta - 1) < \pi, \qquad \eta \in \Gamma$$

We mention several technicalities which must be dealt with but which we gloss over; all of these difficulties are addressed in full in the paper [1]. First, the cut Γ is not closed, since the endpoints are not included. This creates an issue near the points $w = 1, -1$, and one must make delicate arguments in some neighborhoods of these points. Second, throughout this argument we have neglected to concern ourselves with the Hölder continuity of the functional objects with which we are dealing. However one must make sure that certain Hölder conditions are satisfied at each step. In fact the Hölder condition required will **fail** at the endpoints of Γ, but using some advanced theory one can dispense with such issues.

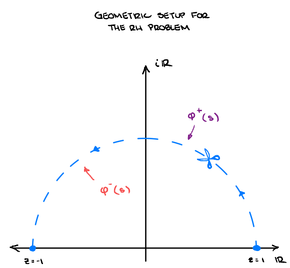

Figure 3: The geometry of the Riemann-Hilbert problem with the indicated cut along the upper half-circle.

3.1 Solving the Riemann-Hilbert Problem

We apply the Sokhotski-Plemelj formula (and in the true sprit of Feynman, ignore a lot of important justifications allowing us to do this) by introducing the Cauchy integral

$$\Omega(s) = \frac{1}{2\pi i} \int_\Gamma \frac{\log G(\omega)}{\omega - s} \, d\omega \tag{3.10}$$

in which case the function $e^{\Omega(s)}$ satisfies the jump condition (3.9). The reader may wonder if we are done, but we are not. Note that any function of the form

$$X_{p,q}(s) = \frac{(s-1)^p}{(s+1)^q} e^{\Omega(s)}$$

also satisfies the jump condition (3.9), and is analytic on $\boldsymbol{C} \setminus \Gamma$. The reason for introducing this function is that to justify the series of transformations to get back to our original problem (the reader will be forgiven for forgetting that we are solving an ODE after all). We need to restrict the growth condition of our solution to the Riemann-Hilbert problem as $|w| \to \infty$ to hope to recover a solution.

Therefore we are now faced with the following problem: choose a branch of the logarithm in (3.10) and parameters p and q so that $\varphi(w) = \mathcal{O}(1/w)$ as $|w| \to \infty$ and $\|\varphi^{\pm}\|_{L^2(\Gamma)} \lesssim 1$.

Following a lengthy technical argument, one finds that the correct choice of p and q are 0 and 1 respectively, which gives us

$$X(s) := X_{0,1}(s) = \frac{1}{s+1} e^{\Omega(s)}$$

so that finally, our solution is given by

$$\varphi(s) = C_0 X(s)$$

where C_0 is an arbitrary constant (that there is an arbitrary constant should not be surprising due to the aforementioned solving of an ODE).

We can now start transforming back into the original-original (two transformations ago) variables. Inverting the gluing function (3.6) and the transformation (3.7) yields

$$\Phi(s) = C_0 Q(s) e^{\frac{1}{3}\pi i(c-s)}$$

where

$$Q(s) = \exp\left[-\frac{i + e^{\frac{2}{3}i\pi(c-s)}}{3} \int_{c-i\infty}^{c+i\infty} \frac{\log G(\tau)}{(1 - e^{\frac{2}{3}i\pi(\tau-c)})(i + e^{\frac{2}{3}i\pi(c-s)})} \, d\tau \right]$$

This allows us to write down an integral (exact) form of our solution, namely

$$f(t) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(s)}{(s-1)(s-2)(s-3)} t^{3-s} \, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Phi(s)}{(s-3)G(s)} t^{3-s} \, ds \tag{3.11}$$

where c is in the analytic strip S.

4 Some Further Results from the Antipov Paper

In my work I am usually concerned more with the transformation of the original problem than actually being able to do computations with the resulting solution. Antipov and Gao [1] spend the latter half of their paper computing series representations and asymptotics of their solution. We will not go through the derivations of these results, but will simply give a survey of the analysis and what it means.

The authors derive the following absolutely convergent series representation for all $x \in \boldsymbol{R}$ (i.e. infinite radius of convergence)

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{6j+2}}{\pi^{2j+2}} \left[\frac{2}{\mu_{6j+2}} \left(\Phi(1)(\nu_{6j+2} - \log x) + \Phi'(1) \right) - \frac{2x^2}{\mu_{6j+4}} \Phi(-1) + \frac{x^3}{\mu_{6j+5}} \Phi(1) \right] \tag{4.1}$$

where μ_k and ν_k are appropriately chosen constants. This series behaves like

$$f(x) \sim -\frac{1}{\pi^2} \Phi(1) x^2 \log x$$

as $x \to 0$ and hence satisfies the first boundary condition (1.3). The series representation is not amenable to determining if the far-field boundary is satisfied, and for this reason we make the following asymptotic expansion of (3.11), valid as $x \to \infty$

$$f(x) \sim 1 + \frac{\Psi(1)}{2x^4} \sum_{j=0}^{\infty} \frac{(-1)^j \mu_{6j+4}}{(3j+2)^2} x^{-6j}$$

This series does not converge (μ is more or less going like a factorial), but with some hand-waving, this is actually fine for representing the behavior as $x \to \infty$. To be more precise about where exactly my hands are waving (and in what language), the more terms of this series you take, the more values of x the series will blow up for. However if you take a finite number of terms, then the resulting function will be valid for large x.

The preceding argument also allows us to fix the arbitrary constant that has been lurking in the shadows for the last few pages. We obtain that

$$C = \frac{e^{-i\pi c/3}}{Q(0)}$$

Finally we have the following plots (from the Antipov paper) which show that the function f will satisfy the two boundary conditions. Figure 4 shows the function f and it's derivatives, computed using the series representation (4.1).

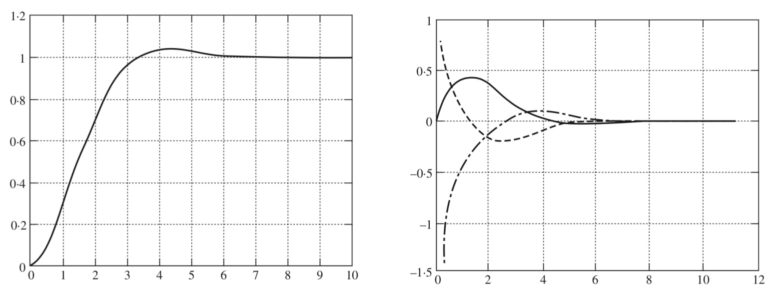

Figure 4: On the left is the function $f(x)$ computed using the series (4.1). The right plots f', f'', and f'''. Taken from Antipov and Gao [1] and modified by the author.

5 Conclusions

Perhaps the reader deserves a fine chocolate or a delicious refreshment[11] after finishing the preceding material. We wish to emphasize a few key points from the exposition above. First, nonlocality is important in physics and is worth studying. Therefore results like the present one play an important role in our understanding of these types of differential equations, even if the method of solution is inelegant. There is hope that preliminary analysis such as the one carried out above will lead to more generalizable solution techniques in the future.

Second is that the techniques arising out of complex analysis are powerful because they give freedom in the ability to transform between domains in a sophisticated way. Because poles and branches are so well studied, singular transformations like (3.6) are dealt with elegantly. The incredible diversity of tools and results which have flourished out of complex analysis highlight it's indispensability in analysis.

The present result also illustrates the fact that sometimes obtaining an explicit solution is only half the battle, since we still cannot do much with the explicit solution itself. We are forced to use asymptotic methods to determine properties of the solution, and even then we may not be able to obtain precise analytic control in a straightforward manner. I have not tried this, but it is conceivably that the series solution (4.1) could be obtained directly by looking at the image of polynomials under the Hilbert transform. If this were the case then the convoluted analysis we have done would be superfluous.

[11] Ideally alcoholic.

5.1 Generalizations to Other Models

Its [4] discusses how converting to Riemann-Hilbert problems is more an art than a science, and that most problems in this vein are attended to by their own bespoke analysis. In the present case we relied fundamentally on the asymptotic properties of our solution, the symmetry of our original equation, the fact that we could write our operator as a Mellin transform, etc., etc. From this perspective, one may think it unlikely that such a method could be generalized.

On the other hand, the Mellin transform is ubiquitous, and the process of converting between the strip S and the complex plane with a cut along the semicircle (or some other curve) is readily generalizable. It is conceivable that a tenacious student could produce a "black box" into which we could place most boundary value problems and obtain solutions.

I originally set out to read this paper to determine if the results were applicable to my research, and after grinding through the pages of computation, I have come to the melancholy conclusion that this method is not what I was looking for. Perhaps in a future project I shall have need for solving an ODE in such a manner, but until then I will content myself with working a level removed from an exact solution.

So long, and thanks for all the fish[12].

A Solution of Grain Boundary Problem via Fourier Methods

We solve (1.3) (without boundary conditions) by the Fourier transform. We take the convention here that

$$\hat{f}(\xi) := \int_{\boldsymbol{R}} f(x) e^{-2\pi i x \xi}\, dx$$

Indeed, taking the Fourier transform of (1.3) gives us the following ODE in the frequency space

$$-\xi \hat{f}_\xi - \hat{f} = \xi^3 \hat{f}$$

where we have effectively **localized** the ODE (1.3) (i.e. the nonlocal equation (1.3) corresponds to a local equation in the frequency space).

The ODE is easily solved using undergraduate ODE techniques and we obtain

$$\hat{f}(\xi) = c\xi e^{-\frac{1}{3}|\xi|^3}$$

Fourier inversion then yields

$$f(x) = c \int_{\boldsymbol{R}} \xi e^{-\frac{1}{3}|\xi|^3 + 2\pi i x \xi}\, d\xi$$

Note that this function does not satisfy the boundary conditions set forth by the original problem (1.3), and hence the authors did not use this (manifestly easier) approach when solving the problem outlined above.

It is possible that this problem can be solved in a more straightforward manner by taking the Fourier transform at the level of the first derivative and integrating the result in space. I have some notes on this process which I have not typed up if anyone is curious. One can do asymptotics on the resulting objects and should get similar results to the ones obtained above by Antipov and Gao.

[12] And by fish we of course mean jokes and complex analysis :)

References

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