PAPER No. T1-3
A Generalization of the Accessibility Problem for Control Systems

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Preliminaries

Let \( M \) be an \( m \times 1 \) dimensional \( C^\infty \) manifold. Points in \( M \) will be denoted by \( p, q, p_0, p, \) etc. Let \( t: M \to \mathbb{R}^m \) be a \( C^\infty \) function such that \( \Delta \neq 0 \). We assume at each \( p \in M \) there exists a neighborhood \( V \) containing \( p \) and a chart \( (t, X): V \to \mathbb{R}^{m+1} \) where \( x = x_1, \ldots, x_m \).

We denote the tangent space to \( M \) at \( p \) by \( T_M p \) and the tangent bundle by \( TM \).

Let \( V \) be a neighborhood of \( p \in M \), we define \( V^+ = \{ q \in V : \tau(q) > \tau(p) \} \) and \( V^- = \{ q \in V : \tau(q) < \tau(p) \} \).

Let \( N \) be another \( C^\infty \) manifold, a function \( f: M \to N \) is \( (p_{\mathcal{W}C^\infty}, C^\infty) \) map if for every \( p \in M \) there exists a neighborhood \( V \) and \( C^\infty \) functions \( g, f: V \to N, i = 1, 2 \), such that \( g_i(q) = f(q) \) for \( q \in V \) and \( g_i(q) = f(q) \) for \( q \in V^- \). We define \( f^+ = f|_{V^+} \) (restricted to \( V^+ \)) and \( f^- = f|_{V^-} \). We consider the function, \( f \), to be double-valued at points \( q \in V \) such that \( \tau(q) \in \tau(p) \). From the context it will be clear which value we mean.

A \((p_{\mathcal{W}C^\infty}, C^\infty)\) vector field, \( X \), is \( p_{\mathcal{W}C^\infty}, C^\infty \) map satisfying

\[
\begin{align*}
(1) & \quad X_p \in T_M p \\
(2) & \quad \langle dt, X \rangle_p = 1
\end{align*}
\]

Each \((p_{\mathcal{W}C^\infty}, C^\infty)\) vector field \( X \) gives rise at least locally to a flow denoted \( \gamma(s)p \), the curve \( s \to \gamma(s)p \) is the solution to the differential equation

\[
\dot{\gamma}(s) = \frac{d}{ds} \gamma(s)p = X_{\gamma(s)p}
\]

with initial condition \( \gamma(0)p = p \).

A vector field system, \( F \), is\( p_{\mathcal{W}C^\infty \to TM} \) (is the collection of all subsets of \( TM \)) satisfying

\[
\begin{align*}
(1) & \quad F_{p} \subseteq T_M p \\
(2) & \quad \text{if } Y_p \in F_p \text{ then } \langle dt, Y \rangle_p = 1
\end{align*}
\]

The vector field system is \( C^\infty \), (alternately \((p_{\mathcal{W}C^\infty}, C^\infty)\)), if for every \( p \in M \) and for every \( \gamma \in F \) there exists \( C^\infty \) (alternately \((p_{\mathcal{W}C^\infty}, C^\infty)\)) vector field \( X \) defined on a neighborhood, \( V \), of \( p \) such that \( X_p = Y_p \) and \( X_q \in F_q \) \( \forall q \in V \). The vector field system, \( F \), is finite or convex if \( \forall p \in M \), \( F_p \) is a finite or convex (respectively) subset of \( TM \). In an abuse of notation we will also use \( F \) to denote \( \{ X : (p_{\mathcal{W}C^\infty}, C^\infty) \text{ vector field and } X_p \in F_p \} \) \( \forall p \in M \). We similarly define \( E \).

Suppose \( X_1, \ldots, X_n \) are \((p_{\mathcal{W}C^\infty}, C^\infty)\) vector fields on \( M \). Henceforth we will use \( E \) to denote the finite vector field system defined by

\[
E_p = \{ X_1p, X_2p, \ldots, X_np \}
\]

and \( F \) to denote the convex vector field system defined by

\[
F_p = \text{convex hull } E_p \subseteq TM_p
\]

If \( X \in E \) and \( A \in F \) then \( X \) will be referred to as an \( E \)-control and \( A \) as an \( F \)-control. \( X \) is also referred to as a bang-bang control.

Accessibility and Controllability.

The set of points \( F \)-accessible from \( p_0 \) is denoted by \( a(F, p_0) \) is defined as \( a(F, p_0) = \{ p \in M : \exists \gamma \in F \text{ with } \gamma \circ \phi \circ a = 0 \text{ and } \gamma(0) = p \} \).

The set of points \( F \)-controllable to \( p_0 \) \( a(-F, p_0) \) is defined as

\[
a(-F, p_0) = \{ p \in M : a(-F, p_0) = \{ p \in M : a(-F, p_0) \subseteq \{ p \in M : \exists A \in F \text{ with } a \circ \phi \circ a = 0 \text{ and } a(0) = p \} \}.
\]

The set of points \( E \)-accessible from \( p_0 \) is denoted by \( a(E, p_0) \) and the set of points \( E \)-controllable to \( p_0 \) \( a(-E, p_0) \) are defined similarly. These sets are sometimes referred to as the set of points bang-bang accessible from \( p_0 \) and bang-bang controllable to \( p_0 \) respectively.

Clearly since \( E \subseteq F \),

\[
a(E, p_0) \subseteq a(F, p_0)
\]

and

\[
a(-E, p_0) \subseteq a(-F, p_0).
\]

Let \( V \) be a neighborhood of \( p_0 \) in \( M \), \( a(F, p_0), \),

\[
\{ p \in V : \exists A \in F \text{ with } a \circ \phi \circ a = 0 \text{ and } a(0) = p \} \in \mathcal{D}(p_0)
\]

The sets \( a(-F, p_0), V \), \( a(E, p_0), V \) and \( a(-E, p_0), V \) are defined accordingly.

Note

\[
a(E, p_0), V \subseteq a(F, p_0), V \subseteq V^+ \subseteq V
\]

and

\[
a(-E, p_0), V \subseteq a(-F, p_0), V \subseteq V^- \subseteq V.
\]

Integrability and Local Semi-integrability

Let \( H: M \to T_M p \)

\[
(1) \quad \text{is a linear subspace of } T_M p, v \in M
\]

\[
(2) \quad v \in M \text{ and } X \in H
\]

there exists a neighborhood, \( V \), of \( p \) and \( C^\infty \) vector field \( X \) defined on \( V \]<= X = Y_p, X \in F_p \), and \( X \notin C^\infty \) V \( q \in V \). \( H \) is called a \( C^\infty \) distribution on \( M \), and we will confuse notation by allowing \( H = \{ X, aC^\infty \text{ vector field} \}

186
field: \( X \subseteq H, \forall p \in M \). \( H \) is non-singular if
\[
p \in H, \forall p \in M \text{ neighborhood of } p \text{ and } C^0 \text{ vector fields } X_1, \ldots, X_n, \text{ such that}
\]
\[
H = \text{span}_\mathbb{R} [X_1, \ldots, X_n] \forall q \in V.
\]

Let \( X, Y \) be \( C^0 \) (alternately \( (\mathfrak{p}w_c^0, C^{\infty}) \)) vector fields on \( M \). The Lie bracket \([X, Y]\) is a \( C^0 \)
(alternately \( (\mathfrak{p}w_c^0, C^{\infty}) \)) vector field on \( M \) defined by
\[
[X, Y] = XY - YX.
\]

Let \( D'H = H + [H, H] \). \( D'H \) is the \( C^0 \) distribution of all linear combinations of vector fields in \( H \) and Lie brackets of vector fields in \( H \) with coefficients from the space of \( C^0 \) real-valued functions on \( M \). We define
\[
D'H = \bigcup_{i=1}^n D'H_i \quad \text{and} \quad DH = \bigcup_{i=1}^n D'H_i.
\]

called the derived system of \( H \).

**Theorem 2.1.** (Frobenius-Hermann [2]).

Suppose \( H \) is a \( C^0 \) distribution on \( M \) such that
on some open neighborhood \( V \) of \( p \), \( H \) satisfies one of the following

(a) \( DH \) is non-singular

(b) \( \dim DH \) of \( \gamma(s) \) is constant along every \( C^0 \) curve \( \gamma(s) \subseteq V \) satisfying \( \gamma(0) = p \), \( \dot{\gamma}(s) \subseteq DH \gamma(t) \)

(c) \( DH \) is locally-finitely generated on \( V \)

(d) \( M \) and \( H \) are real analytic

Then there exists a unique maximal submanifold \( L \), of \( M \) in \( V \) satisfying
\[
p \in L \quad \text{and} \quad \forall q \in L. \quad q \in L.
\]

L is called the integral submanifold of \( H \) (or \( DH \)) in \( V \), through \( p \).

**Corollary 2.2.** Suppose \( E, F \) are \( C^0 \) vector field systems as above and we define a \( C^0 \) distribution \( H \) as
\[
H = \text{span}_\mathbb{R} E \subseteq TM.
\]

\( H \) satisfies the hypothesis of Theorem 2.1 on an open neighborhood \( V \) of \( p_0 \) then
\[
\text{a}(E|p_0, V) \subseteq \text{a}(F|p_0, V) \subseteq L
\]

and
\[
\text{a}(-E|p_0, V) \subseteq \text{a}(-F|p_0, V) \subseteq L.
\]

We would generalize the above result to systems which are only \( (\mathfrak{p}w_c, C^0) \). \( (\mathfrak{p}w_c, C^0) \) distribution, \( H \), on \( M \) is a map
\[
H: M \to TM
\]
satisfying

(i) \( \forall p \in M, \exists \text{ an open neighborhood } V \) of \( p \) such that
\( H \) restricted to \( V^+ \) is a \( C^0 \) distribution on \( V^+ \) and \( H \) restricted to \( V^- \) is a \( C^0 \) distribution on \( V^- \) (we denote the restriction by \( H^+ \) and \( H^- \) respectively).

(ii) \( \exists \varepsilon > 0 \) and \( C^0 \) curves \( \gamma^+: [0, \varepsilon] \to V^+ \)

\( \gamma^-: [0, \varepsilon] \to V^- \) such that
\[
\gamma^+(0) = 0 \quad \text{and} \quad \gamma^-(0) = 0
\]

\( \gamma^+(s) \subseteq H^+ \gamma(s) \)
\( \gamma^-(s) \subseteq H^- \gamma(s) \)
\( \varepsilon > 0 \)

A \( (\mathfrak{p}w_c, C^0) \) distribution \( H \) is locally semi-integrable if for every \( p \in M \), there exists a neighborhood \( V \) of \( M \) and \( C^0 \) restriction \( H^+ \) and \( H^- \), of \( H \) to \( V^+ \) and \( V^- \) such that \( H^+ \) satisfies the hypothesis of Theorem 2.1 on \( V^+ \) and \( H^- \) satisfies the hypothesis on \( V^- \). They need not satisfy the same condition of (a), (b), (c) or (d).

**Theorem 2.3.** Let \( H \) be a locally semi-integrable \((\mathfrak{p}w_c, C^0)\) distribution on \( M \), then at \( p \in M \) there exists a neighborhood \( V \) and unique maximal submanifolds \( L^+ \) and \( L^- \) in \( V^+ \) and \( V^- \) respectively such that
\[
\gamma^+|_{\gamma^+(0)} = DH^+ \gamma(s) \quad \forall q \in V^+
\]
\[
\gamma^-|_{\gamma^-(0)} = DH^- \gamma(s) \quad \forall q \in V^-.
\]

Furthermore, \( p \in L^+ \) and \( p \in L^- \). We call \( L^+ \), \( L^- \) the unique maximal semi-integrable submanifolds of \( H \) (or \( DH \)) through \( p \).

**Corollary 2.4.** Suppose \( E, F \) are \((\mathfrak{p}w_c, C^0)\) vector field systems as above and we define a \((\mathfrak{p}w_c, C^0)\) distribution \( H \) as
\[
H = \text{span}_\mathbb{R} E \subseteq TM
\]

Then if \( H \) is locally semi-integrable there exists a neighborhood \( V \) of \( p_0 \) and semi-integrable submanifolds \( L^+ \) and \( L^- \) of \( H \) such that
\[
\text{a}(E|p_0, V) \subseteq \text{a}(F|p_0, V) \subseteq L^+
\]

and
\[
\text{a}(-E|p_0, V) \subseteq \text{a}(-F|p_0, V) \subseteq L^-.
\]

**REFERENCES**


187