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A Generalization of the Accessibility Problem for Control Systems

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Preliminaries

Let M be an m+1 dimensional C^∞ manifold. Points in M will be denoted by p, q, p_0, p_1, etc. Let t: $M \to R$ be a C^∞ function such that $dt \neq 0$. We assume at each $p \in M$ there exists a neighborhood V containing p and a a chart (t,x): $V \to R^{m+1}$ where $x = x_1,\ldots x_m$.

We denote the tangent space to M at p by T_pM and the tangent bundle by TM.

Let V be a neighborhood of $p \in M$, we define $V^+ = \{q \in V: t(q) > t(p)\}$

and $V^- = \{q \in V: t(q) < t(p)\}$.

Let N be another C^∞ manifold, a function f: $M \to N$ is (pwC^∞,C^∞) map if for every $p \in M$ there exists a neighborhood V and C^∞ functions g_i: $V \to N$, i=1,2, such that $g_1(q) = f(q)$ for $q \in V^+$ and $g_2(q) = f(q)$ for $q = V^-$. We define $f^+ = f\,|_{V^+}$ (f restricted to V^+) and $f^- = f\,|_{V^-}$. We consider the function, f, to be double-valued at points $q \in V$ such that $t(q) \in t(p)$. From the context it will be clear which value we mean.

A (pwC^∞,C^∞) vector field, X, is pw(C^∞,C^∞) map X: $M \to TM$ satisfying

$$\left.\begin{array}{ll}(i) & X_p \in T_pM \\ (ii) & \langle dt,X\rangle p = 1\end{array}\right\} \quad \forall\ p \in M$$

($\langle dt,X$ is the natural pairing between a one-form and a vector field, and in this case $\langle dt,X\rangle p = X(t)p$).

Each (pwC^∞,C^∞) vector field X gives rise at least locally to a flow denoted $\gamma(s)p$, the curve $s \to \gamma(s)p$ is the solution to the differential equation

$$\dot{\gamma}(s)p = \frac{d}{ds}\gamma(s)p = X_{\gamma(s)p}$$

with initial condition

$$\gamma(0)p = p.$$

A vector field system, F, is map F: $M \to 2^{TM}$ (2^{TM} is the collection of all subsets of TM) satisfying

(i) $F_p \subseteq T_pM$ $\qquad \forall\ p \in M$

(ii) if $Y_p \in F_p$ then $\langle dt,Y_p\rangle = 1$

The vector field system is C^∞, (alternately (pwC^∞, C^∞)), if for every $p \in M$ and for every $Y_p \in F_p$ there exists C^∞, (alternately (pwC^∞,C^∞)), vector field X defined on a neighborhood, V, of p such that $X_p = Y_p$ and $X_q \in F_q$, $\forall\ q \in V$. The vector field system, F, is finite or convex if $\forall\ p \in M$, F_p is a finite or convex (respectively) subset of T_pM. In an abuse of notation we will also use F to denote $\{X$: X(pwC^∞,C^∞) vector field and $X_p \in F_p$, $\forall\ p \in M\}$ We similarly define E.

Suppose $X_1,\ldots,X_n$ are (pwC^∞,C^∞) vector fields on M. Henceforth we will use E to denote the finite vector field system defined by

$$E_p = \{X_{1p},X_{2p},\cdots,X_{np}\}$$

and F to denote the convex vector field system defined by

$$F_p = \text{convex hull } E_p \subseteq T_pM$$

If $X \in E$ and $A \in F$ then X will be referred to as an E-control and A as an F-control. X is also referred to as a bang-bang control.

Accessibility and Controlability.

The set of points F-accessible from p_o is denoted by $a(F,p_o)$ is defined as $a(F,p_o) = \{p \in M$: $\exists\ A \in F$ with flow α and $\sigma > 0 \ni p = \alpha(\sigma)p_o\}$.

The set of points F-controllable to p_o $a(-F,p_o) = \{p \in M$: $A \in F$ with flow α and $\sigma > 0 \ni p = \alpha(-\sigma)p_o$ (or $p_o = \alpha(\sigma)p$)$\}$.

The set of points E-accessible from p_o denoted by $a(E,p_o)$ and the set of points E-controllable to p_o denoted by $a(-E,p_o)$ are defined similarly. These sets are sometimes referred to as the set of points bang-bang accessible from p_o and bang-bang controllable to p_o respectively.

Clearly since $E \subseteq F$,

$$a(E,p_o) \subseteq a(F,p_o)$$

and

$$a(-E,p_o) \subseteq a(-F,p_o).$$

Let V be a neighborhood of p_o in M, $a(F,p_o,V) = \{p \in V$: $\exists\ A \in F$ with flow α and $\sigma > 0 \ni p = \alpha(\sigma)p_o$ and $\alpha(s)p_o \in V$ for $s \in [0,\sigma]\}$.

The sets $a(-F,p_o,V)$, $a(E,p_o,V)$ and $a(-E,p_o,V)$ are defined accordingly.

Note

$$a(E,p_o,V) \subseteq a(F,p_o,V) \subseteq V^+ \subseteq V$$

and

$$a(-E,p_o,V) \subseteq a(-F,p_o,V) \subseteq V^- \subseteq V.$$

Integrability and Local Semi-integrability

Let H: $M \to 2^{TM}$ satisfying

(i) H_p is a linear subspace of T_pM, $\forall\ p \in M$

(ii) $\forall\ p \in M$ and $\forall\ Y_p \in H_p$,

there exists a neighborhood, V, of p and C^∞ vector field X such that $X_p = Y_p$ and $X_q \in H_q$, $\forall\ q \in V$. H is called a C^∞ distribution on M, and we will confuse notation by allowing H = $\{X$, a C^∞ vector field: $X_p \in H_p$, $\forall\, p \in M\}$. H is non-singular if $\dim H_p$ = constant. H is locally finitely generated if $\forall$ p M neighborhood V of p and C^∞ vector fields $X_1,\ldots,X_n$ such that

$$H_q = \text{span}\,\{X_{1q},\cdots,X_{nq}\}\ \forall\, q \in V.$$

Let X, Y be C^∞ (alternately (pwC^∞,C^∞)) vector fields on M. The Lie bracket [X,Y] is a C^∞ (alternately (pwC^∞,C^∞)) vector field on M defined by

$$[X,Y] = XY - YX.$$

Let D'H = H + [H,H]. D'H is the C^∞ distribution of all linear combinations of vector fields in H and Lie brackets of vector fields in H with coefficients from the space of C^∞ real-valued functions on M. We define

$$D^iH = D^{i-1}H + [H,D^{i-1}H] \text{ and } DH = \bigcup_{i=1}^{\infty} D^iH.$$ DH is called the derived system of H.

Theorem 2.1. (Frobenius-Hermann [2]).

Suppose H is a C^∞ distribution on M such that on some open neighborhood V of p, H satisfies one of the following

(a) DH is non-singular

(b) $\dim DH_{\gamma(s)}$ is constant along every C^∞ curve $\gamma(s) \in V$ satisfying $\gamma(0) = p$, $\dot\gamma(s) \in DH_{\gamma(t)}$

(c) DH is locally-finitely generated on V

(d) M and H are real analytic

Then there exists a unique maximal submanifold, L, of M in V satisfying

$$p \in L \text{ and } T_qL = DH_q,\ \forall\, q \in L.$$

L is called the integral submanifold of H (or DH) in V, through p.

Corollary 2.2. Suppose E,F are C^∞ vector field systems as above and we define a C^∞ distribution H as

$$H_p = \text{span}\, E_p \subseteq T_pM.$$

H satisfies the hypothesis of Theorem 2.1 on an open neighborhood V of p_o

then

$$a(E,p_o,V) \subseteq a(F,p_o,V) \subseteq L$$

and

$$a(-E,p_o,V) \subseteq a(-F,p_o,V) \subseteq L.$$

We would generalize the above result to systems which are only (pwC^∞,C^∞). A (pwC^∞,C^∞) distribution, H, on M is a map

$$H:\ M \to 2^{TM}$$

satisfying

(i) $\forall\, p \in M$, $\exists$ open neighborhood V of p such that H restricted to V^+ is a C^∞ distribution on V^+ and H restricted to V^- is a C^∞ distribution on V^- (we denote the restriction by H^+ and H^- respectively).

(ii) $\exists\ \varepsilon > 0$ and C^∞ curves $\gamma^+: [0,\varepsilon]\ V^+$ and $\gamma^-: [0,\varepsilon] \to V^-$ such

that $$\gamma^+(0) = \gamma^-(0) = p$$

and $$\dot\gamma^+(s) \in H^+_{\gamma(s)} \qquad s \in (0,\varepsilon)$$
$$\dot\gamma^-(s) \in H^-_{\gamma(s)}$$

A (pwC^∞,C^∞) distribution H is locally semi-integrable if for every $p \in M$, there exists a neighborhood V of M and C^∞ restriction, H^+ and H^-, of H to V^+ and V^- such that H^+ satisfies the hypothesis of Theorem 2.1 on V^+ and H^- satisfies the hypothesis on V^-. They need not satisfy the same condition of (a), (b), (c) or (d).

Theorem 2.3. Let H be a locally semi-integrable (pwC^∞,C^∞) distribution on M, then at $p \in M$ there exists a neighborhood V and unique maximal submanifolds L^+ and L^- in V^+ and V^- respectively such that

$$T_qL^+ = DH^+_q \qquad \forall\, q \in V^+$$
$$T_qL^- = DH^-_q \qquad \forall\, q \in V^-$$

Furthermore, $p \in$ closure L^+ and $p \in$ closure L^-. We call L^+, L^- the unique maximal semi-integrable submanifolds of H (or DH) through p.

Corollary 2.4. Suppose E,F are (pwC^∞,C^∞) vector field systems as above and we define a (pwC^∞,C^∞) distribution H as

$$H_p = \text{span}\, E_p \subseteq T_pM$$

Then if H is locally semi-integrable there exists a neighborhood V of p_o and semi-integrable submanifolds L^+ and L^- of H such that

$$a(E,p_o,V) \subseteq a(F,p_o,V) \subseteq L^+$$

and $$a(-E,p_o,V) \subseteq a(-F,p_o,V) \subseteq L^-.$$

REFERENCES

1. Hermann, Robert, "Differential Geometry and the Calculus of Variations, Academic Press, 241-257, 1968.
2. Hermann, Robert, On the accessibility problem in Control Theory, International Symposium, Nonlinear Differential Equations and Nonlinear Mechanics, 325-332, Academic Press, 1963.