# THE DIFFERENTIAL-GEOMETRIC DUALITY BETWEEN CONTROLLABILITY AND OBSERVABILITY FOR NONLINEAR SYSTEMS

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## 1. INTRODUCTION

The ultimate aim of this work is to put the theory of "observability" for nonlinear systems in a form that is suitable for use in the construction of estimators and observers for complicated nonlinear systems, for example, aircraft. The concepts of Kalman filters and Luenberger observers, that have been so successfully applied to linear systems, have so far received no nice generalization for nonlinear systems, and we believe that geometric ideas might play a crucial role in such a development. This paper is a shorter, more intuitive version of the full-scale one, to be published elsewhere [15], and also reflects a series of talks given by one of us (A.K.) at the Ames conference.

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Our emphasis here is on carrying over to nonlinear systems the "duality" between observability and "controllability" that is so well known and useful for linear systems. We do this not by constructing a "dual" nonlinear system (although that would be a desirable, but harder, approach!) but by exploiting the duality (already well known to differential geometers) between "vector fields" and "differential forms" on manifolds. In turn, the relation between "controllability" and "vector fields" is based on a theorem of W.L. Chow [5,9,10] that has already been extensively exploited, refined, and explained in the control-systems theory literature.

## 2. CONTROLLABILITY AND VECTOR FIELDS

Consider an input-output, continuous-time, state space model of a system of the usual form.

$$\frac{dx}{dt} = f(x,u)$$

$$y = g(x) \quad . \tag{2.1}$$

$x$ is the state vector, element of a space $X$. $u$ is the control vector, element of a space $U$. $y$ is the output vector, element of a space $Y$. Of course, the theory is best known, developed and applied in case it is of the linear form:

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx \qquad (2.2)$$

It is known that "controllability" and "observability" are, in this linear case, respectively the following conditions:

$$B(U),\ AB(U),\ A^2B(U),\ldots \text{ spans the vector space } X \qquad (2.3)$$

$$(\text{kernel } C) \cap \text{kernel}(CA) \cap \text{kernel}(CA^2) \cap \cdots = (0) \qquad (2.4)$$

Look at the nonlinear system (2.1) as living on the manifolds $X$, $U$, $Y$ with all data sufficiently smooth. It was shown in [9] that the condition (2.3) could be generalized in the following way: For each fixed $u$, construct the vector field

$$A_u = f(x,u)\frac{\partial}{\partial x} \qquad (2.5)$$

on $X$. (For the notation of differential geometry, refer to [2] or [10]. In (2.5) we use an obvious "vectorial notation".) Let $V(X)$ denote the Lie algebra (under Jacobi bracket) of smooth vector fields on $X$. Let $\underset{\sim}{L}$ denote the smallest Lie subalgebras of $V(X)$ containing all of the vector fields $A_u$, as $u$ ranges over $U$. $\underset{\sim}{L}$ is called the system Lie algebra, and plays a basic role in many important structural questions concerning nonlinear systems.

For each $x \in X$, let $\underset{\sim}{L}(x) \subset X_x$ denote the linear subspace of the tangent space to $X$ spanned by the values of all the values at $x$ of the vector fields of $\underset{\sim}{L}$.

Definition

The system (2.1) is said to satisfy the controllability rank condition if $\underset{\sim}{L}$ acts transitively on $X$, i.e., if

$$\underset{\sim}{L}(x) \equiv X_x \qquad (2.6)$$

for each $x \in X$ .

Remark. If (2.6) is satisfied, it follows from Chow's theorem [5,10] that, starting at a given point $x$ $X$, every other point is reachable along a solution of (2.1) for some suitable choice of control, but time has to be allowed to run backwards as well as forwards. Partial information about the set reachable by going only forward in time is available from work of Krener [20] and Sussman-Jurdjevich [22].

Another way of thinking of this is that the degree of transitivity of the system Lie algebra $\underset{\sim}{L}$ is an "infinitesimal" measure of "controllability". Our aim now is to give an analogous (and, in fact "dual", in a precise technical technical sense) criterion for "observability".

## 3. THE OBSERVABILITY PFAFFIAN SYSTEM

First, let us review the standard ideas concerning "observability" for systems of type (2.1). (We follow Sussman's ideas [23-26] concerning the manifold formulation of systems of type (2.1).) For each point $x_0$, consider the map

$$\phi_{x_0} : (\text{curves in } U) \rightarrow (\text{curves in } Y)$$

obtained as follows:

> The image of the curve $t \rightarrow u(t)$ is the output curve $t \rightarrow y(t)$ of the system (2.1) with $x_0$ the initial state vector.

Let us say that $x_0, x_1 \in X$ are <u>indistinguishable</u>, written

$$x_0 I x_1 \quad ,$$

if

$$\phi_{x_0} = \phi_{x_1} \quad .$$

I, defined in this way, is an equivalence relation. The system (2.1) is said to be <u>observable</u> if the equivalence classes are just points, i.e., if different points of the state space give different input-output maps.

Let $F(X)$ and $F(Y)$ denote the (commutative, associative, real-scalar) algebra of $C^\infty$, real-valued functions on the manifolds $X$ and $Y$.

$F^1(X)$, $F^1(Y)$ denote the $F(X)$ and $F(Y)$ modules of one-differential forms on $X$ and $Y$. (See [2,10].)

The "read out map"

$$g: X \to Y$$

associated with the system (2.1) will be assumed to be $C^\infty$.

$$g^*: F(Y) \to F(X)$$

$$g^*: F^1(Y) \to F^1(X)$$

denote the dual pull-back map. Let $L \subset V(X)$ denote the Lie subalgebra of vector fields defined by the system (2.1) For each vector field $A \in V(X)$ and one form $\theta \in F^1(X)$, one can define the Lie derivative

$$\mathscr{L}_A(\theta)$$

of $\theta$ by the vector field $A$, as another one-form.

Definition

The <u>observability Pfaffian system</u> $\underset{\sim}{P}$ defined by the system (2.1) is the smallest $F(X)$ submodule of $F^1(X)$

containing the one-forms $g^*(F^1(Y))$ and closed under Lie derivative by vector fields of $\underset{\sim}{L}$. Thus, $\underset{\sim}{P}$ is formed by linear combinations of one-forms of the ~~form~~ type

$$\mathscr{L}_{A_1} \mathscr{L}_{A_2} \cdots (dg^*(y^i)) \quad ,$$

where $y^1,\ldots,y^m$ is a coordinate system for $Y$, and $A_1, A_2, \ldots$ are an arbitrary set of vector fields in the system Lie algebra associated with the system.

If the rank of $\underset{\sim}{P}$ is constant over $X$ (which we shall assume, for simplicity) then it is a <u>completely integrable</u> Pfaffian system, in the Frobenius sense, i.e., the two-forms $d\theta$ for $\theta \in \underset{\sim}{P}$ lie in the Grassman algebra ideal generated by $\underset{\sim}{P}$. Then, by the Frobenius Complete Integrability Theorem [2,10], the maximal submanifolds of $X$ for which all the forms of $\underset{\sim}{P}$ restrict to zero define a <u>foliation</u> of $X$. Our basic observation is that this foliation is (modulo certain possible complications described in more detail in the main paper [15]) just that defined by the indistinguishability equivalence relation $I$. In particular:

> The condition for "observability" which is analogous to the Chow condition for "controllability is that
>
> $$\underset{\sim}{P} = F^1(X) \quad ,$$

i.e., that the values of forms of $\pi^*(F^1(Y))$ and their iterated Lie derivatives under $\underset{\sim}{L}$ have, at each point of $X$, the maximal possible rank.

The full proof of this result is given in [15]. Here, we shall only give a plausibility argument. Let $t \to A^t$ be a one-parameter family of vector fields in $\underset{\sim}{L}$. It may be considered as the infinitesimal generator of a flow on $X$ [10]. The orbits of the flow are the solution curves $t \to x(t)$ of the non-autonomous differential equations

$$\frac{dx}{dt} = A^t(x(t)) \qquad (3.1)$$

Now, suppose that $B$ is a vector field on $X$ which annihilates $\underset{\sim}{P}$, i.e.,

$$\theta(B) = 0 \qquad (3.2)$$

$$\text{for all } \theta \in \underset{\sim}{P} \;.$$

(Geometrically, this condition means that $B$ is tangent to the foliation defined by $\underset{\sim}{P}$.) Let us deform the initial conditions for solutions of (3.1) by means of the orbit curves of $B$. This means that we construct a two-parameter surface

$$(s,t) \to x(s,t)$$

in $X$ such that:

$$\frac{\partial x}{\partial t} = A^t(x(s,t)) \tag{3.3}$$

$$\frac{\partial x}{\partial s}(s,0) = B(x(x,0)) \tag{3.4}$$

Let us now consider a real-valued function $h \varepsilon F(Y)$ on the observation space $Y$. We ask:

> What are the conditions on $A^t$ and $B$ which assure that the function $(s,t) \to h(g(x(s,t))) \equiv h'(s,t)$ <u>does not depend on</u> $s$?

(The relation to the "observability" question for system (2.1) will be explained in a moment.) We can work this out using Lie derivatives:

$$h'(s,t) = g^*(h)(x(s,t))$$

Hence,

$$\frac{\partial h'}{\partial s} = g^*(dh)\left(\frac{\partial x}{\partial s}\right)$$

Hence, using (3.4),

$$\frac{\partial h'}{\partial s}(s,0) = g^*(dh)(B)(x(s,0))$$

$$= 0 \quad , \tag{3.5}$$

since $g^*(dh) \in \underset{\sim}{P}$ and (3.2) is satisfied. Now,

$$\frac{\partial h'}{\partial s \partial t} = g^*(dh)\left(\frac{\partial^2 x}{\partial s \partial t}\right)$$

$$= \frac{\partial}{\partial s}(g^*(dh))(A^t)(x(s,t))$$

Set $t = 0$, and use (3.4):

$$\frac{\partial h'}{\partial s \partial t}(s,0) = d(A^t \lrcorner\, g^*(dh))(B)(x(s,0)) \qquad (3.6)$$

($\lrcorner$ denotes the contraction operation between vector fields and differential forms. See [10].) Now, the fundamental identity linking contraction, Lie derivative and exterior derivative gives:

$$d(A^t \lrcorner\, g^*(dh)) = \mathscr{L}_{A^t}(g^*(dh)) - A^t \lrcorner\, dg^*(dh))$$

$$= \mathscr{L}_{A^t}(g^*(dh)) - \underset{\equiv}{0} .$$

Hence, we can rewrite (3.6) as:

$$\frac{\partial h'}{\partial s \partial t}(s,0) = \mathscr{L}_{A^t}(g^*(dh))(B)(x(s,0))$$

$$= \underset{\equiv}{0} , \qquad (3.7)$$

since B annihilates all differential forms in $g^*(F^1(Y))$ and <u>their arbitrary order Lie derivatives by elements of</u> $\underset{\sim}{L}$.

Continuing in this way, we see that all derivatives with respect to $t$ of $\partial h'/\partial s$ vanish at $t = 0$, hence (at least if all the data is real analytic), that:

$$\frac{\partial h'}{\partial s}(s,t) = 0 \quad , \tag{3.8}$$

i.e., $h'$ is constant in $s$. (In [15], the technical tools for relaxing "real analyticity" are developed.)

Thus, the curves

$$t \to x(s,t)$$

are state-space curves of the system (2.1), with the same input curve, but with differing initial state vector. Hence, (3.8) means that the output

$$y(s,t) = g(x(s,t))$$

is independent of $s$, i.e., the state space curve

$$x(s) \equiv x(s,0)$$

lies in a single equivalence.

To see the system-theoretic interpretation of this result, return to the system in form (2.1). Let $t \to u(t)$ be a curve in the input space $U$. Set:

$$A^t = f(x, u(t)) \frac{\partial}{\partial x} \quad . \tag{3.9}$$

The orbit curves of the flow generated by $A^t$ are the solutions of

$$\frac{dx}{dt} = f(x, u(t)) \quad ,$$

i.e., the state-space curves of the system (2.1) of the "indistinguishability" equivalence relation I. Set:

$$\underset{\sim}{V} = \{B \in V(X) : \underset{\sim}{P}(B) = 0\} \quad .$$

We have then proved that:

> The elements of $\underset{\sim}{V}$ are tangent to the equivalence classes of I.

One can also prove conversely (essentially by just reversing the argument given above), that the values of $\underset{\sim}{V}$ fill up the tangent spaces to the equivalence classes of I, provided that I defines a nonsingular foliation of X. Note that the following properties of $\underset{\sim}{V}$ also hold:

$$[\underset{\sim}{V}, \underset{\sim}{V}] \subset \underset{\sim}{V} \quad .$$

(This is "complete integrability".)

$$[\underset{\sim}{L}, \underset{\sim}{V}] \subset \underset{\sim}{V} \quad ,$$

i.e., the elements of $\underset{\sim}{L}$ <u>leave invariant</u> the foliation $\underset{\sim}{V}$.

Let us now specialize system (2.1) to the familiar linear system (2.2) and show how the "full observability rank conditions", i.e., V = 0, leads to the familiar observability conditions for linear systems.

$$A_u = (Ax + Bu)\frac{\partial}{\partial x}$$

$$dg = C\,dx$$

$$\mathscr{L}_{A_u}(dg) = d(\mathscr{L}_{A_u}(g))$$

$$= d((Ax + Bu)C)$$

$$= (AC)\,dx \quad .$$

$$\mathscr{L}_{A_u}, \mathscr{L}_{A_u}(dg) = d\left((Ax + Bu)\frac{\partial}{\partial x} \lrcorner\ AC\,dx\right)$$

$$= A^2C\,dx \quad .$$

Continuing in this way, we see that the condition that $\underset{\sim}{V} = 0$ is that

$$C,\ AC,\ A^2C,\ \ldots$$

have only zero in the intersection of their kernels.

This way of looking at the observability concept for system (2.1) suggests a more coordinate-free, manifold-theoretic way of <u>defining</u> an input-output system. Suppose manifolds $U, X, Y$ are given, with the following data:

a) A Lie subalgebra $\underset{\sim}{L} \subset V(X)$.

b) A map $u \to A_u$ of $U \to \underset{\sim}{L}$.

c) A "read-out map" $g: X \to Y$ .

One can then identify

"inputs", "states", and "outputs"

in the following way. A curve

$$t \to u(t)$$

in $U$ is an input. It defines a curve $t \to A^t = A_{u(t)}$ in $\underset{\sim}{L}$. Given a state $x_0 \in X$, a curve $t \to x(t)$ is defined as the solution of

$$\frac{dx}{dt} = A^t(x(t)) \equiv A_{u(t)}(x(t)) \quad ,$$

$$x(0) = x_0 \quad .$$

The curve

$$t \to y(t) = g(x(t))$$

is the output. Everything we have done for system (2.1) obviously generalizes to this set-up.

Finally, it is worthwhile pointing out explicitly in what sense the condition "$\underset{\sim}{V} = 0$" we have found for observability is "dual" to the controllability in turn obtained using Chow's theorem. $\underset{\sim}{L}$, as one abstract Lie algebra, acts via derivations on the commutative associative algebra $F(X)$. $V(X)$ is an $F(X)$-module. $\underset{\sim}{L}$ acts--via Jacobi bracket--on $V(X)$, and the Chow controllability criterion involves this action. $F^1(X)$ is the dual $F(X)$-module to $V(X)$. The Lie derivative action of $L^{\cdot}$ in $F^1(X)$ is then just the dual to the Jacobi bracket operation of $L$ in $V(X)$.

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