THE EXISTENCE AND UNIQUENESS OF VOLTERRA SERIES FOR NONLINEAR SYSTEMS

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Abstract

Given an input-output map described by a nonlinear control system $\dot{x} = f(x,u)$ and nonlinear output $y = h(x)$, we present a simple straightforward means for obtaining a series representation of the output $y(t)$ in terms of the input $u(t)$. When the control enters linearly $\dot{x} = f(x) + ug(x)$ the method yields the existence of a Volterra series representation. The proof is constructive and explicitly exhibits the kernels. It depends on standard mathematical tools such as the Fundamental Theorem of Calculus and the Cauchy estimates for the Taylor series coefficients of analytic functions. In addition, the uniqueness of Volterra series representations is discussed.

1. Introduction

Consider a control system Σ of the general form

$$(1.1)\qquad \begin{aligned} \dot{x} &= f(x,u) \\ x(0) &= x^0 \\ y &= h(x) \end{aligned}$$

where the input takes values in $\mathbb{R}^\ell$, the state x is an element of $\mathbb{R}^m$, and the output y takes values in $\mathbb{R}^n$. The vector field f and output function h are assumed to possess a sufficient degree of smoothness. Depending on the form of f, the input function u(t) is either absolutely integrable on [0,T] or is bounded and measurable. In other words u belongs to one of the two Banach spaces $L^1([0,T],\mathbb{R}^\ell)$ or $L^\infty([0,T],\mathbb{R}^\ell)$. In either case the output y(t) is a member of the Banach space $C^0([0,T],\mathbb{R}^n)$ for each choice of input. In view of this it is natural to associate with Σ the <u>input-output map</u>

$$\psi: L^r([0,T],\mathbb{R}^\ell) \to C^0([0,T],\mathbb{R}^n)$$

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where $r = 1,\infty$.

For the linear system

$$(1.2)\qquad \begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x^0 \\ y &= Cx \end{aligned}$$

where A, B, C are constant matrices of appropriate dimension, the map ψ is affine and given by the variation of constants formula

$$(1.3)\qquad y(t) = Ce^{At}x^0 + \int_0^t Ce^{A(t-s)}Bu(s)ds.$$

The kernel function $w(t) = Ce^{At}B$ is often referred to as the impulse response function or the weighting pattern of the system.

If the system is not linear then ψ does not have such a simple representation as (1.3). Wiener, in the 1940's, gave consideration to representation of ψ in a form similar to

$$(1.4)\quad y(t) = w_0(t) + \int_0^t w_1(t,s)h(s)ds + \int_0^t\int_0^{s_1} w_2(t,s_1,s_2)u(s_2)u(s_1)ds_2ds_1 + \dots,$$

a generalization of (1.3). Functional expansions such as (1.4) were first considered by Volterra [9] and have since become known as Volterra series. Since their introduction by Wiener, papers dealing with Volterra series have appeared periodically in the system theory literature although until recently questions regarding validity and/or convergence of such a series have been treated lightly.

In part, the current interest in the Volterra series representation stems from a study of bilinear systems

$$(1.5)\qquad \begin{aligned} \dot{x} &= Ax + \sum_{i=1}^{\ell} u_iB_ix \\ x(0) &= x^0 \\ y &= Cx. \end{aligned}$$

In their paper [3] d'Alessandro, Isidori, and Ruberti showed that bilinear systems do possess

Volterra expansions and explicit formulas for the calculation of the kernel functions $w_k(t,s_1,\ldots,s_k)$ were given. Brockett [1] used a technique of Carleman [2] together with an approximation result of Krener [7] to show existence and uniqueness of Volterra series for systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{\ell} u_i g(x)$$

$$x(0) = x^0 \tag{1.6}$$

$$y = h(x)$$

where the functions f, g, and h are analytic. Recently Gilbert [4] has developed similar results for (1.1) and (1.6) based on functional analytic techniques due to Graves [5]. The major purpose for the present paper is to introduce a simpler and more straightforward approach to results similar to those of Brockett and Gilbert.

2. The Basic Algorithm

For the sake of simplicity, we consider (1.6) for scalar input and scalar output. The case of a vector input and a vector output offers no conceptual difficulty.

Let $\gamma_0(t,s,x)$ be the solution of the differential equation,

$$\frac{d}{dt}\gamma_0(t,s,x) = f(\gamma_0(t,s,x))$$

such that

$$\gamma_0(s,s,x) = x.$$

Given $u \in L^1(0,T],\mathbb{R}')$, denote by $\gamma_u(t,s,x)$ the solution of

$$\frac{d}{dt}\gamma_u(t,s,x) = f(\gamma_u(t,s,x)) + u(t)g(\gamma_u(t,s,x))$$

satisfying

$$\gamma_u(s,s,x) = x.$$

Further, we designate points of the form $\gamma_0(s,0,x^0)$, $\gamma_u(s,0,x^0)$ by the simpler notation $x_0(s)$, $x_u(s)$ respectively.

Next, consider the locus, $\rho(s)$, of end-points obtained via concatenation of the trajectory $x_u(\tau)$, $0 \le \tau \le s$, with the trajectory $\gamma_0(\tau,s,x_u(s))$, $s \le \tau \le t$. The curve is given by

$$\rho(s) = \gamma_0(t,s,x_u(s)) = \gamma_0(t-s,0,x_u(s))$$

and satisfies

$$\rho(0) = x_0(t), \quad \rho(t) = \gamma_u(t).$$

Applying the Fundamental Theorem of Calculus to the function $h(\rho(s))$ yields

$$h(x_u(t)) = h(\gamma_0(t)) + \int_0^t \frac{d}{ds}h(\rho(s))ds. \tag{2.1}$$

To calculate the above integrand we first define the function $H(s_1,s_2)$ by

$$H(s_1,s_2) = h(\gamma_0(t-s_1,0,x_u(s_2)).$$

Then,

$$h(\rho(s)) = H(s,s),$$

$$\frac{d}{ds}h(\rho(s)) = \left(\frac{\partial H_1}{\partial s_1} + \frac{\partial H_2}{\partial s_2}\right)(s,s).$$

This last equation in conjunction with the equation,

$$\left.\frac{\partial h(\gamma_0(t-s,0,x))}{\partial x}f(x)\right|_{x=x_u(s)} = \left.\frac{\partial h(x)}{\partial x}f(x)\right|_{x=\gamma_0(t-s,0,x_u(s))},$$

(γ_0 is the flow of f) yields

$$\frac{d}{ds}h(\rho(s)) = u(s)\left.\frac{\partial h(\gamma_0(t-s,0,x))}{\partial x}g(x)\right|_{x=x_u(s)}$$

$$= u(s)\left.\frac{\partial h(\gamma_0(t,s,x))}{\partial x}g(x)\right|_{x=x_u(s)}.$$

In view of this, equation (2.1) becomes

$$h(x_u(t)) = w_0(t) + \int_0^t \bar{w}_1(t,s,x_u(s))u(s)ds, \tag{2.2}$$

where

$$w_0(t) = h(x_0(t))$$

$$\bar{w}_1(t,s,x) = \frac{\partial h(\gamma_0(t,s,x))}{\partial x}g(x).$$

Next, we replace $h(\cdot)$ by $\bar{w}_1(t,s,\cdot)$ and use (2.2) to compute $\bar{w}_1(t,s,x_u(s))$ obtaining

$$\bar{w}_1(t,s,x_u(s)) = w_1(t,s) + \int_0^s \bar{w}_2(t,s,r,x_u(r))u(r)dr,$$

where

$$w_1(t,s) = \bar{w}_1(t,s,x_0(s))$$

$$\bar{w}_2(t,s,r,x) = \frac{\partial \bar{w}_1(t,s,\gamma_0(s,r,x))}{\partial x} g(x).$$

Substitution of this into (2.2) yields

$$h(x_u(t)) = w_0(t) + \int_0^t w_1(t,s)u(s)ds$$
$$+ \int_0^t \int_0^s \bar{w}_2(t,s,r,x_u(r))u(r)u(s)drds.$$

After k repetitions of the above we have $h(x_u(t))$ represented as

$$h(x_u(t)) = w_0(t) + \sum_{m=1}^{k-1} \int_0^t w_m(t,s^m)u_m(s^m)ds^m$$
$$+ \int_0^t \bar{w}_k(t,s^k x(s_k))u_k(s^k)ds^k$$

where the expression,

$$\int_0^t w_m(t,s^m)u_m(s^m)ds^m,$$

denotes the multi-integral,

$$\int_0^t \cdots \int_0^{s_{m-1}} w(t,s_1,\ldots,s_m)u(s_1)\ldots u(s_m)ds_m \ldots ds_1.$$

The functions w_m, $\bar{w}_m$ are given recursively by the formulas,

$$\bar{w}_m(t,s^m,x) = \frac{\partial \bar{w}_{m-1}(t,s^{m-1},\gamma_0(s_m,s_{m-1},x))}{\partial x} g(x)$$

(2.3)

$$w_m(t,s^m) = \bar{w}_m(t,s^m,x_0(s_m)).$$

Continuing indefinitely, the procedure outlined here yields a formal series of the type (1.4). For bilinear systems with output map $h(x) = x$ the above reduces to Picard iteration. However, in the general case the two are not equivalent.

Sufficient hypotheses on f,g,h which justify the above developments can be given. Definitions and theorems pertaining to these matters occur in the next section. In addition, uniqueness of such expansions is also treated. The proofs of these results as well as the extension of the basic algorithm to systems of the form (1.1) are to appear elsewhere.

3. Definitions and Statements of Theorems

The first three theorems in the section pertain to the case where Σ has the form (1.6) while the last considers the question of uniqueness of a Volterra representation. Preceding these are the pertinent definitions. We continue to assume scalar input and scalar output. Definitions and theorem statements could be worded so as to accommodate vector inputs and outputs, but this is not pursued since it does not introduce any conceptual difficulties but does produce considerable notational complications.

(3.1) <u>Definition</u> Let ψ_1, ψ_2 be maps from $L^1([0,T],\mathbb{R})$ into $C^0([0,T],\mathbb{R})$ and u, $u_0 \in L^1([0,T],\mathbb{R})$. By the notation

$$\psi_1(u) = \psi_2(u) + o(\|u - u_0\|_1^k)$$

we mean that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|\psi_1(u) - \psi_2(u)\|_0 < \varepsilon\|u - u_0\|_1^k$$

whenever

$$\|u - u_0\|_1 < \delta.$$

Here $\|\cdot\|_1$ and $\|\cdot\|_0$ denote the usual norms on $L^1([0,T],\mathbb{R})$ and $C^0([0,T],\mathbb{R})$. Similarly,

$$\psi_1(u) = \psi_2(u) + \mathcal{O}(\|u - u_0\|_1^k)$$

if for some $\varepsilon > 0$ there exists a $\delta > 0$ such that the above inequality holds.

(3.2) <u>Definition</u> ψ has a <u>Volterra expansion of length-k</u> if there are k + 1 functions (kernels) $w_0, w_1, \ldots, w_k$ such that,

(i) w_0 is defined on $[0,T]$, w_m is defined on $\{(t,s_1,\ldots,s_m) | 0 \le s_m \le \cdots \le s_1 \le t \le T\}$.

(ii) Each w_i is continuous on its domain of definition.

(iii) $\psi(u)(t) = w_0(t) + \sum_{m=1}^{k} \int_0^t w_m u_m ds^m + o(\|u\|_1^k)$

(3.3) <u>Definition</u> ψ has a <u>Volterra series representation</u> if there exists a set of kernels w_m, $0 \le m < \infty$, satisfying (i), (ii) and there exists a $\delta > 0$ such that whenever $\|u\|_1 < \delta$

$$\psi(u)(t) = w_0(t) + \sum_{m=1}^{\infty} \int_0^t w_m(t,s^m)u_m(s^m)ds^m.$$

The series converges in norm topology on $C^0([0,T],\mathbb{R})$ for all $\|u\|_1 < \delta$.

(3.4) <u>Theorem</u> Let f and g be C^1 vector fields on $\mathbb{R}^m$. Given that the initial value problem

$$\dot{x} = f(x), \quad x(0) = x^0$$

has a solution on $[0,T]$, then for some fixed $\delta > 0$

$$\dot{x} = f(x) + ug(x), \quad x(0) = x^0$$

has a solution on $[0,T]$ for all u with $\|u\|_1 < \delta$.

(3.5) <u>Theorem</u> Let $h \in C^{k+1}$, $f \in C^{k+1}$, $g \in C^k$, $k \ge 1$. If the initial value problem

$$\dot{x} = f(x), \quad x(0) = x^0$$

is solvable on $[0,T]$, then ψ has a Volterra expansion of length-k.

(3.6) Theorem Let f,g be analytic vector fields and h an analytic function. If $\dot{x} = f(x)$, $x(0) = x^0$ has a solution on $[0,T]$, then ψ has a Volterra series representation.

(3.7) Theorem Let ψ_1, ψ_2 be input output maps which have Volterra expansions of length-k. If $\psi_1 = \psi_2 + o(\|u\|_1^k)$ then $w_m^1 = w_m^2$, $0 \le m \le k$, where w_m^j is the m^{th} kernel of ψ_j, $j = 1,2$.

(3.8) Corollary If an input output map ψ has a Volterra series representation then it is unique.

The proof of Theorem (3.4) depends on the following:

(3.9) Lemma Let $\mu(s) \ge 0$, $\nu(s) \ge 0$, $\mu, \nu \in L^1[0,T]$.
If f is continuous on $[0,T]$ and satisfies

$$0 \le f(s) \le \int_0^s [\mu(t)f(t) + \nu(t)dt$$

for all $s \in [0,T]$ then

$$0 \le f(s) \le e^{\int_0^s \mu(r)dr} \int_0^T \nu(r)dr.$$

The following estimate, which is based on the wellknown Cauchy estimates of analytic function theory, is central to the proof of (3.6).

(3.10) Lemma Let $G_1, G_2, \ldots, G_\ell, h$ be analytic functions of n variables and have modulus bounded by M on the closed polydisc P with center a and radii $r_j = R$. Then,

$$\left|\left(G_\ell \frac{\partial}{\partial z_{i_\ell}}\right)\cdots\left(G_1 \frac{\partial}{\partial z_{i_1}}\right)h(a)\right| \le \ell!\left(\frac{2^{n+1}M}{R}\right)^\ell M.$$

4. Concluding Remarks

The principle aim of the preceding has been to offer a simpler way of arriving at the Volterra series representation for the input output map of (1.6). Recursive equations, (2.3), permit a geometric description of the m^{th} kernel in the Volterra Series. It is this interpretation which leads to the proof of convergence via standard tools and results of analysis.

For the case of bilinear systems the explicit kernel formulas in [3] are easily obtained from equations (2.3). In the general case, (1.6), we follow along the lines of [1]. Explicit formulas are obtained by utilizing the approximation results of [7] together with equations (2.3).

5. References

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