A Note on Commutative Bilinear Optimal Control

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Abstract—**For optimal control problems with bilinear dynamics and quadratic cost, the assumption of commutativity is extremely strong and reduces the problem to the classical one of least action.**

In a recent paper [1] Wei and Pearson treated an optimal control problem with commutative bilinear dynamics and quadratic cost. The purpose of this note is to point out that commutativity reduces the problem to a classical one of least action. It is also equivalent to the problem of shortest distance in Euclidean space and hence the solutions are straight lines, i.e., curves of constant velocity (control).

Consider the bilinear system

$$\dot{x} = \left(B_0 + \sum_{i=1}^{m} B_i u_i\right) x \qquad (1)$$

$$x(t_0) = x_0$$

as in [1]. Here, x is an n vector, $B_0, \cdots, B_m$ are constant $n \times n$ matrices and $u_1(t), \cdots, u_m(t)$ are assumed square integrable on $[t_0, t_1]$. If we assume that $B_0, \cdots, B_m$ commute, i.e., $B_i B_j = B_j B_i, \forall i,j$; then it is easy to show that

$$e^{B_i s_i} e^{B_j s_j} = e^{B_j s_j} e^{B_i s_i} = e^{B_i s_i + B_j s_j} \qquad (2)$$

and

$$e^{B_i s_i} B_j = B_j e^{B_i s_i}. \qquad (3)$$

From (2) it is clear that the set of accessible points of (1) (excluding x_0) is precisely the set of all x such that

$$x = e^{B_m s_m} \cdots e^{B_1 s_1} e^{B_0 s_0} x_0 \qquad (4)$$

where $s_0, s_1, \cdots, s_m \in \Re$ and $s_0 > 0$.

Suppose $\{B_0 x_0, \cdots, B_k x_0\}$ is a maximal linearly independent set drawn from $\{B_0 x_0, \cdots, B_m x_0\}$ (possibly after reordering). Using (2) and (3) it is easy to show that set of accessible points is all x such that

$$x = e^{B_k s_k} \cdots e^{B_1 s_1} e^{B_0 s_0} x_0 \qquad (5)$$

where $s_0, \cdots, s_m \in \Re$ and $s_0 > 0$. Let M be the range of (5) for $s_0, \cdots, s_m \in \Re$, then M is a $(k+1)$-dimensional submanifold of $\Re^n$ and a homogeneous space of abelian group action generated by $B_0, \cdots, B_k$. This implies that M is a product of circles and lines. Actually, (5) defines $s = (s_0, \cdots, s_k)$ as local coordinates on M. In these new coordinates (1) becomes the very simple system:

$$\dot{s}_0 = 1$$
$$\dot{s}_1 = u_1$$
$$\vdots$$
$$\dot{s}_k = u_k. \qquad (6)$$

In these coordinates the second problem of [1], to minimize

$$J_2(x_1) = \min_u \int_{t_0}^{t_1} u' R u \, dt$$

subject to (1) and $x(t_1) = x_1$, (or $s(t_0) = s_0$, $s(t_1) = s_1$) is the classical problem of least action. It is well known that the solution is a curve of constant velocity (control), i.e., a straight line in the s coordinates.

The first problem of [1], to minimize $J_1(x_1) = x_1' F x_1 + J_2(x_1)$ subject to (1) must also have solutions of constant control found by first minimizing J_1 with respect to x_1, and then J_2 with respect to u.

References

[1] K. C. Wei and A. E. Pearson, "On minimum energy control of commutative bilinear systems," this issue, pp. 1020–1023.

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