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CONTINUOUS LINEAR PROGRAMMING
AND PIECEWISE BILINEAR SYSTEMS

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1. INTRODUCTION

Linear programming is one of the most useful applied mathematical tools developed in the last thirty years. The problem of extremizing a linear functional over a convex subset of R^n defined by a set of linear inequalities arises naturally in many diverse fields. Such problems admit both elegant mathematics (the duality theory) and an efficient algorithm for their solution (the simplex method).

Since 1956 attempts have been made to generalize linear programming to infinite dimensional spaces. Not only is this a natural mathematical extension but, more importantly, there are numerous potential applications. Unfortunately the situation is much more complicated and only limited successes have been achieved. Most of the effort has been in extending the duality theory (see [1-7] and their references); very little effort has been devoted to extending the simplex method [8, 9]. This of course is natural, for the latter depends very heavily on the former, but from an applications point of view a computationally feasible algorithm is more important. However, numerous simple examples have been solved using a simplex-like algorithm. This leads one to hope that a machine implementable algorithm might some day be available for certain broad classes of infinite dimensional linear programs. We might add in passing that such problems have also been called continuous linear programs, generalized linear programs and bottleneck problems.

The mathematical tools most frequently employed in studying infinite dimensional linear programs have been functional analysis and convex analysis. In particular the separating hyperplane theorem (Hahn-Banach theorem) has played a crucial role. This requires the consideration of convex sets with nonempty interior and, for reasons that we shall discuss later, has been the major difficulty in extending the finite dimensional duality results.

The purpose of this paper is to propose an alternate approach based on variable structure systems and optimal control theory. A certain class of infinite dimensional linear programs can be viewed as piecewise bilinear optimal control problems and the duality theory of such programs is closely connected with the Pontryagin Maximum Principle. Hopefully a "cross pollination" can lead to progress in both fields.

The rest of the paper is organized as follows. Section 2 introduces a very simple example illustrating the class of problems under discussion. Section 3 discusses the dual program and Section 4 the literature on duality. In Section 5 a simplex-like method is used to solve the example of Section 2 and in the last section we discuss the relationship with variable structure systems.

2. AN EXAMPLE

We start by considering a simple example of an infinite dimensional linear program after R. S. Lehman [8]. It deals with a one sector economic model.

Let $x_1(t)$ be the amount of steel stockpiled at time t and $x_2(t)$ be the amount of steelmaking capacity available at time t. We normalize so that one unit of capacity can produce one unit of steel in one unit of time.

Let $z_1(t)$ be the rate of steel production and $z_2(t)$ be the rate of production of steelmaking capacity. If we assume that α units of steel can instantaneously be converted to one unit of steelmaking capacity then the dynamics is

$$\dot{x}_1 = z_1 - \alpha z_2$$
$$\dot{x}_2 = z_2$$

subject to initial conditions

$$x_1(0) = c_1 \qquad x_2(0) = c_2$$

and the constraints

$$x_i(t) \geq 0 \quad , \quad z_i(t) \geq 0$$
$$x_2(t) \geq z_1(t).$$

Our goal is to maximize the steel stockpile at some terminal time $T > 0$.

It is interesting to note that this problem is very close to that considered by George Oster elsewhere in this volume. The wasp queens correspond to stockpiled steel and the wasp workers to steelmaking capacity.

One can rewrite the steelmaking problem in a different fashion, namely,

$$\max \int_0^T z_1(t) - \alpha z_2(t)\, dt \tag{2.1}$$

subject to

$$\int_0^t \alpha z_2(s) - z_1(s)\, ds \leq c_1$$
$$z_1(t) - \int_0^t z_2(s)\, ds \leq c_2$$
$$z_i(t) \geq 0 \qquad i=1,2.$$

By the addition of slack variables it can also be written as

$$\max \int_0^T z_1(t) - \alpha z_2(t)\, dt$$

subject to

$$z_3(t) + \int_0^t \alpha z_2(s) - z_1(s)\, ds = c_1 \tag{2.2}$$

$$z_1(t) + z_4(t) - \int_0^t z_2(s)\, ds = c_2 \tag{2.3}$$

$$z_i(t) \geq 0 \qquad i=1,2,3,4.$$

Formulated in this fashion the steel problem appears as an infinite dimensional version of the finite dimensional linear programming problem. In the next section we formulate a general class of problems to which this example belongs.

3. DUAL PROGRAMS

Consider the problem of finding $z(t) = (z_1(t),\ldots,z_n(t))^t$ which maximizes the integral

$$\int_0^T a(t)z(t)\, dt \tag{3.1}$$

subject to the constraints

$$z(t) \geq 0 \tag{3.2}$$

$$B(t)z(t) \leq c(t) + \int_0^t K(t,s)z(s)\, ds. \tag{3.3}$$

Such problems are called continuous linear programs [8], they are not the most general infinite dimensional linear program but they do include many interesting cases.

We shall refer to this as the primal problem. If $z(t)$ satisfies (3.2) and (3.3) it is called feasible, if in addition (3.3) is strictly satisfied then $z(t)$ is strictly feasible. The supremum of (3.1) over all feasible $z(t)$ is denoted by P, if this is achieved by a feasible $z(t)$ then such a solution is called optimal. The program is autonomous if $a(t)$, $c(t)$, $B(t)$ and $K(t,s)$ are constant functions of t and s.

From the applications which motivate the formulation of problems of the above type, we are accustomed to think of $z(t)$, $a(t)$, $c(t)$, $B(t)$ and $K(t,s)$ as vector or matrix valued functions satisfying some sort of regularity condition, i.e., piecewise continuous or integrable. In particular, since $z(t)$ plays the role of a control or decision variable, we would certainly wish to allow it to be piecewise continuous. On the other hand, one is faced with the problem of existence of feasible and optimal solutions. To ensure this in some problems one might wish to consider $z_i(t)$ as living in a larger space of (generalized) functions. For our purposes, we shall assume that $z_i(t) \in Z$, a locally convex space which includes the piecewise continuous functions. Possible choices which have been considered in the literature include $L^P[0,T]$, $1 \leq P \leq \infty$, the space of Borel measures on $[0,T]$ and various spaces of generalized functions (distributions). Let $Z^{n\times 1}$ denote the space of $n\times 1$ vectors

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We deliberately leave Z unspecified. Instead we take as part of the problem the task of finding a dual pair (Z,W) of locally convex spaces in which (3.1), (3.2) and (3.3) make sense, such that an optimal solution exists and strong duality holds. We elaborate on this.

The inequalities (3.2) and (3.3) are defined via the cone Z_+ of <u>nonnegative elements</u> of Z. For function spaces this is the cone of functions which are non-negative almost everywhere. For generalized function spaces, this is the dual cone to the space of nonnegative test functions.

Let Z and W be a dual pair, i.e., W is a locally convex space of continuous linear functionals on Z which is big enough to separate points of Z. Usually W is taken as the topological dual of Z and frequently Z is assumed to be reflexive, i.e., the topological dual of W. However, one does not need to assume this, for if Z and W are given the weak topologies, then they are the duals of each other, hence reflexive. Let $W^{1\times n}$ denote the space of $1\times n$ vectors of elements of W.

One consideration in the choice of Z and W is that (3.1) defines a continuous linear functional on Z, $a(t) \in W^{1\times n}$. The integral denotes the natural pairing between $W^{1\times n}$ and $Z^{n\times 1}$. Other considerations are that the map

$$L: z(t) \mapsto B(t)z(t) - \int_0^t K(t,s)z(s)\,ds \tag{3.4}$$

be continuous from $Z^{n\times 1}$ to $Z^{m\times 1}$ relative to the weak topologies induced by $W^{1\times n}$ and $W^{1\times m}$, and that $c(t) \in Z^{m\times 1}$.

As in finite dimensional linear programming, an inequality constraint can be converted to an equality constraint by the introduction of a nonnegative slack variable. Also an equality constraint can be replaced by a pair of inequalities.

Another similarity is the existence of a dual problem to (3.1), (3.2), (3.3), namely that of finding $w(t) \in W^{1\times m}$ to minimize

$$\int_0^T w(t)c(t)\,dt \tag{3.5}$$

subject to

$$w(t) \geq 0 \tag{3.6}$$

and

$$w(t)B(t) \geq a(t) + \int_t^T w(s)K(s,t)\,ds. \tag{3.7}$$

The inequalities are defined relative to the dual cone W_+ of Z_+ and (3.7) is defined by the dual of L,

(3.8)
$$L^*: W^{1\times m} \to W^{1\times n}$$
$$L^*: w(t) \mapsto w(t)B(t) - \int_t^T w(s)K(s,t)\, ds.$$

L^* exists and is weakly continuous if L is continuous with respect to the weak topologies on Z and W.

The notation of (3.7) and (3.8) is suggested by the construction of the adjoint when all the objects are functions and the integrals make sense.

Feasible and optimal solutions to the dual problem are defined as before. We denote by D the infimum of (3.5) over all feasible $w(t)$. The dual is specifically constructed so that weak duality holds, $P \le D$. The proof is immediate, if $z(t)$ and $w(t)$ are feasible then

(3.9)
$$\int_0^T a(t)z(t)\, dt \le \int_0^T \left(w(t)B(t) - \int_t^T w(s)K(s,t)\, ds\right) z(t)\, dt$$
$$= \int_0^T w(t)\left(B(t)z(t) - \int_0^t K(t,s)z(s)\, ds\right) dt$$
$$\le \int_0^T w(t)c(t)\, dt.$$

If one chooses to work abstractly using $\langle \cdot,\cdot \rangle$ to denote dual pairings then (3.9) becomes

(3.10)
$$\langle a,z\rangle \le \langle L^*(w),z\rangle$$
$$= \langle w,L(z)\rangle$$
$$\le \langle w,c\rangle.$$

The final considerations in the choice of Z and W are that strong duality hold, $P = D$, and there exist optimal solutions to both problems. It is still an open question whether there always exists such a choice of Z and W even for autonomous problems.

A pair of feasible solutions, $z(t)$ and $w(t)$, have complementary slackness if

(3.11)
$$\langle w,\ c-L(z)\rangle = 0$$
$$\langle L^*(w)-a,\ z\rangle = 0.$$

It follows from (3.10) that if $z(t)$ and $w(t)$ are optimal with $P = D$ then they have complementary slackness. On the other hand, complementary slackness for feasible solutions implies strong duality and the optimality of both.

If $z(t)$ and $w(t)$ are integrable functions which are optimal, then there is a stronger form of complementary slackness [5] for almost all t such that $z_j(t) > 0$.

(3.12)
$$w(t)B_{\cdot j}(t) - \int_t^T w(s)K_{\cdot j}(s,t)\, ds = a_j(t)$$

where $B_{\cdot j}$ and $K_{\cdot j}$ deno
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where $B_{\cdot j}$ and $K_{\cdot j}$ denote the j^{th} columns of B and K. Also for almost all t such that $w_i(t) > 0$

$$B_{i\cdot}(t)z(t) - \int_0^t K_{i\cdot}(s,t)z(s)\,ds = c_i(t) \tag{3.13}$$

where $B_{i\cdot}$ and $K_{i\cdot}$ denote the i^{th} rows of B and K.

4. STRONG DUALITY

In finite dimensions if both the primal and dual programs are feasible then both are optimal with $P = D$. For Z and W fixed a priori this need not be true, for a counterexample see Grinold [2].

Surprisingly enough this difficulty does not arise per se from the jump to infinite dimensions. If the inequalities of a finite dimensional linear program are replaced by inclusions into convex cones then such a problem is called a generalized linear program and similar difficulties can occur [3].

The root of the problem is that some of the convex sets associated to the program lack interior. A frequent approach taken by several authors is to restrict to a class of problems where the sets have interior, then the Hahn-Banach theorem can be used to prove strong duality and the existence of optimal solutions to both the primal and dual programs. The technique is well known, we formulate a particular application of it in the following theorem.

Theorem. Suppose that the primal is strictly feasible, the dual is feasible, and Z_+ has nonempty interior, then the dual program has an optimal solution and $P = D$.

(Note: The assumption of strict feasibility is frequently called a Slater condition.)

Proof. From feasibility and weak duality

$$-\infty < P \leq D < \infty .$$

Define a pair of convex sets in $\mathbb{R} \times Z^{m\times 1}$ by

$$E_1 = \{(\alpha,\beta) : \exists\, z \geq 0 \ni \alpha \leq \langle a,z\rangle - P,\ \beta \leq c - L(z)\}$$

$$E_2 = \{(\alpha,\beta) \geq 0\}.$$

Since Z_+ has nonempty interior so do E_1 and E_2. The interiors must be disjoint else there exists a strictly feasible z such that $\langle a,z\rangle > P$. By the Hahn-Banach theorem there exists a nontrivial $(v,w) \in \mathbb{R} \times W^{1\times m}$ separating E_1 and E_2, i.e., for every $(\alpha_i,\beta_i) \in E_i$

$$v\alpha_1 + \langle w,\beta_1\rangle \leq v\alpha_2 + \langle w,\beta_2\rangle . \tag{4.1}$$

The range of cone E_2 with vertex 0 under the linear functional (v,w) is a cone with vertex 0 in $\mathbb{R}$. Since it is bounded below by (4.1) it must either be 0 or the cone of nonnegative reals. This shows that

$$(v,w) \geq 0.$$

Since $(0,0) \in E_2$, for every $z \geq 0$

$$v \langle a,z \rangle - vP + \langle w,c \rangle - \langle w,Lz \rangle \leq 0.$$

This implies that

(4.2) $$\langle va-L^*(w),\ z \rangle \leq 0$$

and

(4.3) $$\langle w,c \rangle \leq vP.$$

Suppose $v = 0$, then (4.2) and (4.3) imply that

(4.4) $$L^*(w) \geq 0$$

(4.5) $$\langle w,c \rangle \leq 0.$$

If (4.5) is strict and y is any feasible solution for the dual problem, then from (4.4) $y + \epsilon w$ is also feasible for all $\epsilon \geq 0$. As $\epsilon \to \infty$, $\langle y+\epsilon w,\ c \rangle \to -\infty$ so $D = -\infty$, a contradiction. If (4.5) is an equality then choose a strictly feasible z. The set of all β such that

$$0 < \beta < c - L(z)$$

is a nonempty open set since Z_+ has interior. From (4.1) it follows that

$$\langle w,\beta \rangle \leq 0$$

for all such β. On the other hand, since $w \geq 0$ and $\beta > 0$, it is true that

$$\langle w,\beta \rangle \geq 0.$$

Therefore w annihilates a nonempty open set implying $w = 0$. This contradicts the nontriviality of (v,w).

From the preceding paragraph we conclude that $v \neq 0$, hence it can be normalized to 1. Inequality (4.4) implies that w is feasible for the dual problem and inequality (4.5) and weak duality imply that it is optimal, $D = \langle w,c \rangle = P$. QED

For those problems where Z_+ has empty interior or the Slater condition fails, Duffin [1] has introduced an asymptotic approach. A sequence $\{(z^k,y^k)\} \subseteq Z^{n \times 1} \times Z^{m \times 1}$ is feasible if

(4.6) $$z^k \geq 0 \quad , \quad y^k \geq 0$$

and

(4.7) $$\lim_k (c-L(z^k)-y^k) = 0.$$

The value of the program for such a sequence is

(4.8) $$\lim_k \sup \langle a,z^k \rangle$$

Duffin defines the <u>subvalue</u> SP of linear program (3.1), (3.2) and (3.3) to be the supremum of (4.6) over all feasible sequences and has shown that if both the primal and dual are feasible then SP = D.

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The introduction of feasible sequences in effect thickens the positive cone in the constraint space (the range of L) and ensures that the Slater condition is satisfied for a sequence of perturbed problems. This allows the employment of the Hahn-Banach theorem to obtain a sequence of feasible solutions to the dual problem whose value (3.5) converges to D.

Another approach to strong duality is found in the work of Tyndall [2,4], Levinson [3], and Grinold [5]. These authors impose inequality restrictions on the matrices $B(t)$, $K(t,s)$ and the vectors $a(t)$, $c(t)$ which must be satisfied for each t and s. These restrictions are considerably stronger than requiring primal and dual feasibility. They then discretize the time variable to approximate the infinite dimensional programs by finite dimensional programs. The latter are feasible from the inequality assumptions and, under some additional regularity assumptions, they show that the optimal solutions of the finite dimensional programs converge to optimal solutions of the infinite dimensional programs as the time step goes to 0.

5. THE SIMPLEX METHOD

What makes finite dimensional linear programming important is the existence of an efficient algorithm, the simplex method, for computing solutions. The rudiments of a similar algorithm exist in infinite dimensions but one could hardly call it well defined at present. Following Lehman [8] we illustrate the algorithm using the steelmaking example of Section 2 using for convenience the operational calculus notation. Drews, Hartberger and Segers [9] contains similar examples.

We introduce the matrix P whose entries are distributions

$$P = \begin{pmatrix} -1_+ & \alpha 1_+ & \delta & 0 \\ \delta & -1_+ & 0 & \delta \end{pmatrix}$$

where δ is the Dirac delta function and 1_+ is the Heavyside function,

$$1_+(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

We use $*$ to denote the convolution product of (generalized) functions with support in $[0,\infty)$

$$f * z_i(t) = \int_0^\infty f(t-s) z_i(s)\, ds.$$

In particular

$$\delta * z_i(t) = \int_0^\infty \delta(t-s) z_i(s)\, ds = z_i(t)$$

$$1_+ * z_i(t) = \int_0^\infty 1_+(t-s) z_i(s)\, ds$$
$$= \int_0^t z_i(s)\, ds$$

Convolution with f is a linear transformation from Z into itself. The adjoint maps W into W and is given by

$$w_i \circ f(t) = \int_0^\infty w_i(s) f(s-t)\, ds.$$

Notice that $*$ is commutative while $\circ$ is not.

We express (2.2) and (2.3) in this notation as

(5.1) $$P * z(t) = c(t).$$

Subject to this constraint and

(5.2) $$z(t) \geq 0$$

we wish to maximize

$$\int_0^T a(t) z(t)\, dt.$$

Here

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}, \quad c(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad a(t) = (a_1, a_2, a_3, a_4) = (1, -\alpha, 0, 0)$$

The dual problem is to find $w(t) = (w_1(t), w_2(t))$ such that

(5.3) $$w \circ P(t) \geq a(t)$$

which minimizes

(5.4) $$\int_0^T w(t) c(t)\, dt.$$

Mimicking the simplex method, we choose an initial basis consisting of $z_3(t)$ and $z_4(t)$. Let $\hat{z}(t)$ denote basic variables of $z(t)$ and $\hat{P}(t)$ the corresponding columns of $P(t)$, then (5.1) determines a solution

(5.5) $$\hat{z}(t) = \hat{P}^{-1} * c.$$

We can invert square matrices whose entries are distributions as long as the determinant computed using convolution multiplication is invertible. In this case it is particularly simple since $\hat{P}(t)$ is the identity matrix, hence its own inverse.

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The resulting solution is

$$(5.6) \qquad z(t) = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ c_2 \end{pmatrix}$$

which is feasible since $c_1 \geq 0$.

To test the optimality of (5.6) we determine a $w(t)$ using complementary slackness, i.e.,

$$\hat{a}(t) - w \circ \hat{P}(t) = 0.$$

If this is feasible then (5.5) is optimal,

$$(5.7) \qquad \begin{aligned} w(t) &= \hat{a} \circ \hat{P}^{-1}(t) \\ w(t) &= (0,0). \end{aligned}$$

Define $\pi(t)$ by

$$(5.8) \qquad \begin{aligned} \pi(t) &= a(t) - w \circ P(t) \\ \pi(t) &= (1,-\alpha,0,0). \end{aligned}$$

If $\pi(t) \leq 0$ then w is dual feasible. If one component of $\pi(t)$ is positive at some time (as is $\pi_1(t)$) we wish to introduce the corresponding component of $z(t)$ into the basis. To see which variable leaves we look at the equation

$$\hat{P} * \hat{z}(t) = c - z_1 * P_{.1}(t)$$

where $P_{.1}$ is the first column of P. Multiplying by $\hat{P}^{-1}$ yields the inequality

$$(5.9) \qquad \hat{z}(t) = \hat{P}^{-1} * c(t) - z_1 * \hat{P}^{-1} * P_{.1}(t) \geq 0$$

$$\begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - z_1 * \begin{pmatrix} -1+ \\ \delta \end{pmatrix} \geq 0$$

If $z_1 \geq 0$ the first inequality is always satisfied so it is z_4 which is dropped from the basis. Our new basis is

$$z(t) = \begin{pmatrix} z_1(t) \\ z_3(t) \end{pmatrix}.$$

As before we solve (5.5) to obtain the new solution

$$(5.10) \qquad z(t) = \begin{pmatrix} c_2 \\ 0 \\ c_1 + c_2 t \\ 0 \end{pmatrix}$$

We check optimality by computing $w(t)$ and $\pi(t)$ from (5.7) and (5.8)

$$w(t) = (0,1)$$

$$\pi(t) = (0,(T-t)-\alpha,0,-1).$$

If $T \leq \alpha$ then $\pi(t) \leq 0$ on $[0,T]$ so $w(t)$ is dual feasible and (5.10) is the optimal solution.

If $T > \alpha$ then we wish to introduce $z_2(t)$ into the basis on the interval $[0,T-\alpha]$ where $\pi_2(t) \geq 0$. The new basis changes with time, it is z_2 and either z_1 or z_3 from $[0,T-\alpha]$ and is z_1 and z_3 from $[T-\alpha,T]$. We decide which of z_1 or z_3 to drop on the first interval by the analog of (5.9)

$$\begin{pmatrix} z_1(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} c_2 \\ c_1 + c_2 t \end{pmatrix} - z_2 * \hat{P}^{-1} * P_{\cdot 2} \geq 0$$

$$= \begin{pmatrix} c_2 \\ c_1 + c_2 t \end{pmatrix} - z_2 * \begin{pmatrix} -1_+ \\ \alpha 1_+ - t 1_+ \end{pmatrix} \geq 0.$$

The first inequality is satisfied for all $z_2(t) \geq 0$ so we drop $z_3(t)$. This turns out to be the optimal solution and involves a delta function.

We have chosen a very simple example to illustrate the method, we refer the reader to [8] and [9] for much more difficult ones.

One unpleasant thing that can happen is that the basis solution could involve delta functions or derivatives of delta functions. Of course the latter are not nonnegative, hence not feasible, but the former are. Another difficulty of this method is knowing over which interval to introduce a new basis variable. One choice is the interval where $\pi_i(t)$ is positive, but this is not always the right one. Also we should mention that it is possible for a variable to drop out of the basis at one time only to be replaced by another at a different time [9].

These difficulties notwithstanding, this loosely-defined algorithm judiciously employed, does seem to work. It would be immensely useful if it could be rigorously defined and if some kind of convergence established.

6. PIECEWISE BILINEAR CONTROL

In this section we discuss the relationship between autonomous Continuous Linear Programs and Variable Structure Systems. Consider the program of maximizing

(6.1) $$\int_0^T az(t)\,dt$$

subject to the constraints

(6.2) $$z(t) \geq 0$$

and

(6.3)

Such a problem is s…
singular. Hencefor…
From the above…
state variable x = …
maximize $x_0(T)$ wher…

(6.4)

and

(6.5)

Differentiating we…

(6.6)

and

(6.7)

Let $\beta = \{j_1, \ldots$
is the j_i^{th} column…
matrix whose j_i^{th} …
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m of $\{1,\ldots,n\}$. …

Let $b_1,\ldots,b_\ell$ …
It can be shown th…

(6.8)

where

(6.9)

Combining (6.…

(6.10)

$$\text{(6.3)} \qquad Bz(t) = c + \int_0^t Kz(s)\, ds.$$

Such a problem is said to be <u>nonsingular</u> if every mxm submatrix of B is nonsingular. Henceforth we shall assume nonsingularity.

From the above we define an optimal control problem of the Mayer type with state variable $x = (x_0, x_1)$ where x_0 is a scalar and x_1 is an mX1 vector. We wish to maximize $x_0(T)$ where

$$\text{(6.4)} \qquad x_0(t) = \int_0^t az(t)\, dt$$

and

$$\text{(6.5)} \qquad x_1(t) = c + \int_0^t Kz(s)\, ds.$$

Differentiating we obtain

$$\text{(6.6)} \qquad \dot{x}_0 = az$$

and

$$\text{(6.7)} \qquad \dot{x}_1 = Kz.$$

Let $\beta = \{j_1, \ldots, j_m\} \subseteq \{1, \ldots, n\}$ and let B_β be an mxm matrix whose i^{th} column is the j_i^{th} column of B. It is nonsingular by assumption. Let B_β^+ be the nxm matrix whose j_i^{th} row is the i^{th} row of B_β^{-1} and whose other rows are zero. We refer to β as a basis and use it as an index ranging over all subsets of cardinality m of $\{1, \ldots, n\}$. From (6.3) and (6.5) a basis β is <u>feasible</u> at x if

$$z = B_\beta^+ x_1 \geq 0.$$

Let $b_1, \ldots, b_\ell$ be the generators of the extreme rays of set $\{z \geq 0 : Bz = 0\}$. It can be shown that if $z \geq 0$ and $Bz = x_1$ then there exists u, v such that

$$\text{(6.8)} \qquad z = \sum_{\beta \text{ feasible}} u_\beta B_\beta^+ x_1 + \sum_{j=1}^{\ell} v_j b_j$$

where

$$\text{(6.9)} \qquad \sum_{\beta \text{ feasible}} u_\beta = 1, \quad u_\beta \geq 0, \quad \text{and} \quad v_j \geq 0.$$

Combining (6.6), (6.7), (6.8) and (6.9) we obtain

$$\text{(6.10)} \qquad \dot{x}_0 = \Sigma u_\beta a B_\beta^+ x_1 + \Sigma v_j a b_j$$

(6.11) $$\dot{x}_1 = \Sigma u_\beta K B_\beta^+ x_1 + \Sigma v_j K b_j$$

This is a <u>piecewise</u> <u>bilinear</u> <u>system</u> in the following sense. Let C_β be the convex cone defined by

$$C_\beta = \{x_1 : B_\beta^+ x_1 > 0\}.$$

Consider the class of sets with nonempty interior formed by intersections and complements of the C_β's, call the minimal elements of this class $\mathcal{D}_1,\dots,\mathcal{D}_n$. They are atoms of the algebra generated by the C_β's. In general $\mathcal{D}_j$ is not a cone but it is closed under multiplication by positive scalars. When x_1 is restricted to a particular $\mathcal{D}_j$ the collection of feasible bases does not change, so on $\mathcal{D}_j$ the dynamics (6.10) and (6.11) is bilinear in the usual sense.

This representation of a linear program as an optimal control problem is quite natural. Suppose z(t), an integrable function, is primal optimal and x(t) and u(t), v(t) are defined by (6.4), (6.5) and (6.8). By the Pontryagin Maximum Principle there exists a dual variable $\lambda(t) = (\lambda_0(t), \lambda_1(t))$ which is closely related to the dual of the linear program. If w(t) is an integrable function which is dual optimal and $x(t) \in \bigcup_j \mathcal{D}_j$ for almost all $t \in [0,T]$ then

(6.12) $$\lambda_1(t) = \int_t^T w(s)\, ds.$$

To see this, we define the Hamiltonian

(6.13)
$$\begin{aligned} H(\lambda,x,u,v) &= \lambda_0 \dot{x}_0 + \lambda_1 \dot{x}_1 \\ &= (\lambda_0 a + \lambda_1 K) z(x,u,v) \\ &= (\lambda_0 a + \lambda_1 K)(\sum_\beta u_\beta B_\beta^+ x_1 + \sum_j v_j b_j). \end{aligned}$$

The Maximum Principle states that if u(t), v(t) and x(t) are optimal, there exists a $\lambda(t) = (\lambda_0(t), \lambda_1(t))$ satisfying the adjoint differential equations

(6.14) $$\dot{\lambda}_0 = -\frac{\partial H}{\partial x_0} = 0$$

(6.15) $$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = (\lambda_0 a + \lambda_1 K) \sum_\beta u_\beta B_\beta^+$$

the transversality conditions,

(6.16) $$\lambda_0(T) \geq 0 \quad , \quad \lambda_1(T) = 0$$

and for any admissible controls u, v

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On the other hand,

(6.17) $$H(\lambda(t),x(t),u,v) \leq H(\lambda(t),x(t),u(t),v(t)).$$

On the other hand, if w(t) is dual optimal then complementary slackness requires that for almost all t such that $z_j(t) > 0$,

$$w(t)B_{\cdot j} = a_j + \int_t^T w(s)K_{\cdot j}\, ds.$$

If $x(t) \in \mathcal{D}_i$ and $u_\beta(t) > 0$, then $z_j(t) > 0\ \forall\ j \in \beta$, so

$$w(t)B_\beta = a_\beta + \int_t^T w(s)\, ds\, K_\beta$$

or equivalently

$$w(t) = (a + \int_t^T w(s)\, ds\, K)B_\beta^+.$$

We multiply by $u_\beta(t)$ and sum over β to obtain

(6.18) $$w(t) = (a + \int_t^T w(s)\, ds\, K) \sum_\beta u_\beta B_\beta^+.$$

Define $\lambda_0(t) = 1$ and $\lambda_1(t) = \int_t^T w(s)\, ds$, then $\lambda(T)$ satisfies the transversality conditions and (6.18) implies that for almost every t, $\lambda(t)$ satisfies the adjoint differential equation. Finally w(t) is dual feasible so

(6.19) $$w(t)B \geq a + \int_t^T w(s)\, ds\, K.$$

Given any admissible u, v, we define

$$\xi = \Sigma u_\beta B_\beta^+ x_1 + \Sigma v_j b_j$$

then

$$B\xi = x_1 \quad \text{and} \quad \xi \geq 0$$

so

$$\begin{aligned} H(\lambda(t),x(t),u,v) &= (a + \int_t^T w(s)\, ds\, K)\xi \\ &\leq w(t)B\xi \\ &= w(t)x'. \end{aligned}$$

On the other hand, by complementary slackness

$$w(t)x' = w(t)Bz(t) = (a + \int_t^T w(s)\, ds\, K)z(t)$$

so (6.17) is satisfied.

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ABSTRACT

The purpose of th
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1. INTRODUCTION

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