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CONTINUOUS LINEAR PROGRAMMING
AND PIECEWISE BILINEAR SYSTEMS

ARTHUR J. KRENER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, DAVIS, CA. 95616, USA

1. INTRODUCTION

Linear programming is one of the most useful applied mathematical tools developed in the last thirty years. The problem of extremizing a linear functional over a convex subset of $\mathbb{R}^n$ defined by a set of linear inequalities arises naturally in many diverse fields. Such problems admit both elegant mathematics (the duality theory) and an efficient algorithm for their solution (the simplex method).

Since 1956 attempts have been made to generalize linear programming to infinite dimensional spaces. Not only is this a natural mathematical extension but, more importantly, there are numerous potential applications. Unfortunately the situation is much more complicated and only limited successes have been achieved. Most of the effort has been in extending the duality theory (see [1-7] and their references); very little effort has been devoted to extending the simplex method [8, 9].

This of course is natural, for the latter depends very heavily on the former, but from an applications point of view a computationally feasible algorithm is more important. However, numerous simple examples have been shown using a simplex-like algorithm. This leads one to hope that a machine implementable algorithm might some day be available for certain broad classes of infinite dimensional linear programs. We might add in passing that such problems have also been called continuous linear programs, generalized linear programs and bottleneck problems.

The mathematical tools most frequently employed in studying infinite dimensional linear programs have been functional analysis and convex analysis. In particular the separating hyperplane theorem (Hahn-Banach theorem) has played a crucial role. This requires the consideration of convex sets with nonempty interior and, for reasons that we shall discuss later, has been the major difficulty in extending the finite dimensional duality results.

The purpose of this paper is to propose an alternate approach based on variable structure systems and optimal control theory. A certain class of infinite dimensional linear programs can be viewed as piecewise bilinear optimal control problems and the duality theory of such programs is closely connected with the Pontryagin Maximum Principle. Hopefully a "cross pollination" can lead to progress in both fields.

The rest of the paper is organized as follows. Section 2 introduces a very simple example illustrating the class of problems under discussion. Section 3 discusses the dual program and Section 4 the literature on duality. In Section 5 a simplex-like method is used to solve the example of Section 2 and in the last section we discuss the

We start by considering a steelmaking process after R. S. Lehman. Let $x_1(t)$ be the amount of steelmaking capacity $x_2(t)$ at time $t$. Let $z_1(t)$ be the amount of steelmaking capacity $z_2(t)$ at time $t$. We find $z_1(t)$ to be subject to initial conditions and the constraints.

Our goal is to maximize the profit. It is interesting to observe the relationships between steel and the wasp work. We can rewrite this as

$$\text{(2.1)}$$

subject to

$$\text{(2.2)}$$

By the addition of slack variables, we can rewrite this as

$$\text{(2.3)}$$

subject to

$$\text{(2.4)}$$
section we discuss the relationship with variable structure systems.

2. AN EXAMPLE

We start by considering a simple example of an infinite dimensional linear program after R. S. Lehman [8]. It deals with a one sector economic model.

Let \( x_1(t) \) be the amount of steel stockpiled at time \( t \) and \( x_2(t) \) be the amount of steelmaking capacity available at time \( t \). We normalize so that one unit of capacity can produce one unit of steel in one unit of time.

Let \( z_1(t) \) be the rate of steel production and \( z_2(t) \) be the rate of production of steelmaking capacity. If we assume that \( \alpha \) units of steel can instantaneously be converted to one unit of steelmaking capacity then the dynamics is

\[
\begin{align*}
\dot{x}_1 &= z_1 - \alpha z_2, \\
\dot{x}_2 &= z_2
\end{align*}
\]

subject to initial conditions

\[
x_1(0) = c_1, \quad x_2(0) = c_2
\]

and the constraints

\[
x_1(t) \geq 0, \quad x_2(t) \geq z_1(t).
\]

Our goal is to maximize the steel stockpile at some terminal time \( T > 0 \).

It is interesting to note that this problem is very close to that considered by George Oster elsewhere in this volume. The wasp queens correspond to stockpiled steel and the wasp workers to steelmaking capacity.

One can rewrite the steelmaking problem in a different fashion, namely,

\[
(2.1) \quad \max \int_0^T (z_1(t) - \alpha z_2(t)) \, dt
\]

subject to

\[
\begin{align*}
\int_0^T \alpha z_2(s) - z_1(s) \, ds &\leq c_1, \\
z_1(t) - \int_0^T z_2(s) \, ds &\leq c_2, \\
z_1(t) &\geq 0, \quad i=1,2.
\end{align*}
\]

By the addition of slack variables it can also be written as

\[
(2.2) \quad \max \int_0^T z_3(t) - \alpha z_2(t) \, dt
\]

subject to

\[
z_3(t) + \int_0^T \alpha z_2(s) - z_1(s) \, ds = c_1
\]
3

\[(2.3)\]
\[z_1(t) + z_4(t) - \int_0^t z_2(s) \, ds = c_2\]
\[z_i(t) \geq 0 \quad i=1,2,3,4.\]

Formulated in this fashion the steel problem appears as an infinite dimensional version of the finite dimensional linear programming problem. In the next section we formulate a general class of problems to which this example belongs.

3. DUAL PROGRAMS

Consider the problem of finding \( z(t) = (z_1(t), \ldots, z_n(t))^T \) which maximizes the integral

\[(3.1)\]
\[\int_0^T a(t)z(t) \, dt \]

subject to the constraints

\[(3.2)\]
\[z(t) \geq 0\]

\[(3.3)\]
\[b(t)z(t) \leq c(t) + \int_0^t k(t,s)z(s) \, ds.\]

Such problems are called continuous linear programs [8], they are not the most general infinite dimensional linear program but they do include many interesting cases.

We shall refer to this as the primal problem. If \( z(t) \) satisfies (3.2) and (3.3) it is called feasible, if in addition (3.3) is strictly satisfied then \( z(t) \) is strictly feasible. The supremum of (3.1) over all feasible \( z(t) \) is denoted by \( P \).

If this is achieved by a feasible \( z(t) \) then such a solution is called optimal. The program is autonomous if \( a(t), c(t), b(t) \) and \( k(t,s) \) are constant functions of \( t \) and \( s \).

From the applications which motivate the formulation of problems of the above type, we are accustomed to think of \( z(t), a(t), c(t), b(t) \) and \( k(t,s) \) as vector or matrix valued functions satisfying some sort of regularity condition, i.e., piecewise continuous or integrable. In particular, since \( z(t) \) plays the role of a control or decision variable, we would certainly wish to allow it to be piecewise continuous. On the other hand, one is faced with the problem of existence of feasible and optimal solutions. To ensure this in some problems one might wish to consider \( z_i(t) \) as living in a larger space of (generalized) functions. For our purposes, we shall assume that \( z_i(t) \in Z \), a locally convex space which includes the piecewise continuous functions. Possible choices which have been considered in the literature include \( L^p[0,T] \), \( 1 \leq p \leq \infty \), the space of Borel measures on \([0,T]\) and various spaces of generalized functions (distributions). Let \( Z^{nx1} \) denote the space of \( nx1 \) vectors of elements of \( Z \) then.

We deliberately leave the task of finding \( a \) and (3.3) make sense, so we elaborate on this.

The inequalities (elements of \( Z \). For \( k(t,s) \) negative almost everywhere to the space of nonnegative

Let \( Z \) and \( W \) be various linear functionals \( W \) is taken as the top sensitive, i.e., the top of this, for if \( Z \) and \( W \) are each other, hence reflexive of \( W \).

One consideration of linear functional between \( W^{nxm} \) and \( Z^{nxm} \).

\[(3.4)\]
\[be continuous from \( Z^{nxm} \rightarrow W^{nxm} \), and that \( c(t) \in Z^{nx1}.\]

As in finite dim-

converted to an equal-

variable. Also an equ-

Another similarly namely that of finding

\[(3.5)\]
\[subject to\]

\[(3.6)\]
\[and\]

\[(3.7)\]

The inequalities are by the dual of \( L,\)
The inequalities are defined relative to the dual cone $Z^*$ of $Z$, and (3.2) is defined

\[ \forall x \in X, \exists ! y \in Y, \forall z \in Z, y(z) = \langle x, z \rangle \]

subject to

\[ \forall x \in X, \exists ! y \in Y, \forall z \in Z, y(z) = \langle x, z \rangle \]

and

\[ \forall x \in X, \exists ! y \in Y, \forall z \in Z, y(z) = \langle x, z \rangle \]

As in finite-dimensional linear programming, an inequality constraint can be

\[ \forall x \in X, \exists ! y \in Y, \forall z \in Z, y(z) = \langle x, z \rangle \]

converted to an equality constraint by the introduction of a nonnegative slack

\[ \forall x \in X, \exists ! y \in Y, \forall z \in Z, y(z) = \langle x, z \rangle \]

variable. Also an equality constraint can be replaced by a pair of inequalities.

Another similarity is the existence of a dual problem to (3.1), (3.2), (3.3).

The integrals denote the standard pairing.

\[ \forall x \in X, \exists ! y \in Y, \forall z \in Z, y(z) = \langle x, z \rangle \]

(3.4)
\[ L^* : w(t) - \int_T^0 w(s)K(s,t) \, ds. \]

(3.8)

where \( B_j \) and \( K_j \) denote such that \( w_j(t) > 0 \)

(3.13)

In finite dimension, both are optimal with for a counterexample:

Surprisingly enough, finite dimensions. It is replaced by inclusion of linear programs.

The root of the program lack interior to a class of problems. It can be used to prove the primal and dual parallel application of it.

**Theorem.** Suppose \( Z_p \) has nonempty interior \( P = D \).

(4.1)

**Proof.** From feasibility.

Define a pair of convex

Since \( Z_p \) has no interior, there exists a theorem there exists every \( (z_j, B_j) \in E_k \)

(4.1)

The range of \( c \) with vertex 0 in \( R_n \).
where $B_j$ and $K_j$ denote the $j$th columns of $B$ and $K$. Also for almost all $t$ such that $w_j(t) > 0$

$$B_j(t)z(t) = \int_0^t K_j(s,t)z(s) \, ds = c_j(t)$$

(3.13)

where $B_j$ and $K_j$ denote the $i$th rows of $B$ and $K$.

4. STRONG DUALITY

In finite dimensions if both the primal and dual programs are feasible then both are optimal with $P = D$. For $Z$ and $W$ fixed a priori this need not be true, for a counterexample see Grinold [2].

Surprisingly enough this difficulty does not arise per se from the jump to infinite dimensions. If the inequalities of a finite dimensional linear program are replaced by inclusions into convex cones then such a problem is called a generalized linear program and similar difficulties can occur [3].

The root of the problem is that some of the convex sets associated to the program lack interior. A frequent approach taken by several authors is to restrict to a class of problems where the sets have interior, then the Hahn-Banach theorem can be used to prove strong duality and the existence of optimal solutions to both the primal and dual programs. The technique is well known, we formulate a particular application of it in the following theorem.

Theorem. Suppose that the primal is strictly feasible, the dual is feasible, and $Z_+$ has nonempty interior, then the dual program has an optimal solution and $P = D$.

(Note: The assumption of strict feasibility is frequently called a Slater condition.)

Proof. From feasibility and weak duality

$$P \leq D \leq \infty.$$

Define a pair of convex sets in $\mathbb{R} \times \mathbb{R}^{m \times 1}$ by

$$E_1 = \{ (\sigma, \beta) : \exists z \geq 0 \exists \sigma \leq \langle a, z \rangle - P, \beta \leq c - L(z) \}$$

$$E_2 = \{ (\sigma, \beta) \geq 0 \}.$$

Since $Z_+$ has nonempty interior so do $E_1$ and $E_2$. The interiors must be disjoint else there exists a strictly feasible $z$ such that $\langle a, z \rangle > P$. By the Hahn-Banach theorem there exists a nontrivial $(v, w) \in \mathbb{R} \times \mathbb{R}^{m \times 1}$ separating $E_1$ and $E_2$, i.e., for every $(\sigma_1, \beta_1) \in E_1$

$$v \sigma_1 + \langle w, \beta_1 \rangle \leq v \sigma_2 + \langle w, \beta_2 \rangle.$$

The range of cone $E_2$ with vertex 0 under the linear functional $(v, w)$ is a cone with vertex 0 in $\mathbb{R}$. Since it is bounded below by (4.1) it must either be 0 or the
cone of nonnegative reals. This shows that

\[ (v, v) \geq 0. \]

Since \((0,0) \in E_2\), for every \(z \geq 0\)

\[ v(a, z) - vP + (w, c) - (w, Lz) \leq 0. \]

This implies that

\[ \langle va - L^*(w), z \rangle \leq 0 \]
and

\[ \langle w, c \rangle \leq vP. \]

Suppose \(v = 0\), then (4.2) and (4.3) imply that

\[ L^*(w) \geq 0 \]
and

\[ \langle w, c \rangle \leq 0. \]

If (4.5) is strict and \(y\) is any feasible solution for the dual problem, then from (4.4) \(y + cv\) is also feasible for all \(c \geq 0\). As \(c \to \infty\), \(\langle y + cv, w \rangle \to \infty\), so \(D = \infty\), a contradiction. If (4.5) is an equality then choose a strictly feasible \(z\). The set of all \(\beta\) such that

\[ 0 < \beta < c - L(z) \]

is a nonempty open set since \(Z_+\) has interior. From (4.1) it follows that

\[ \langle w, \beta \rangle \leq 0 \]
for all such \(\beta\). On the other hand, since \(v \geq 0\) and \(\beta > 0\), it is true that

\[ \langle w, \beta \rangle \geq 0. \]

Therefore \(w\) annihilates a nonempty open set implying \(w = 0\). This contradicts the nontriviality of \((v, w)\).

From the preceding paragraph we conclude that \(v \neq 0\), hence it can be normalized to 1. Inequality (4.4) implies that \(w\) is feasible for the dual problem and inequality (4.5) and weak duality imply that it is optimal, \(D = \langle w, c \rangle = P\). QED

For those problems where \(Z_+\) has empty interior or the Slater condition fails, Duffin [1] has introduced an asymptotic approach. A sequence \(\{x^k, y^k\} \subseteq \mathbb{R}^{n \times 1} \times \mathbb{R}^{m \times 1}\) is feasible if

\[ x^k \geq 0, \quad y^k \geq 0 \]
and

\[ \lim_{k \to \infty} (c - L(x^k) - y^k) = 0. \]

The value of the program for such a sequence is

\[ \lim \sup_{k} \langle a, x^k \rangle \]

Duffin defines the subvalue SP of linear program (3.1), (3.2) and (3.3) to be the supremum of (4.6) over all feasible sequences and has shown that if both the primal and dual are feasible then \(SP = D\).
The introduction of feasible sequences in effect thickens the positive cone in the constraint space (the range of $L$) and ensures that the Slater condition is satisfied for a sequence of perturbed problems. This allows the employment of the Hahn-Banach theorem to obtain a sequence of feasible solutions to the dual problem whose value $(3.5)$ converges to $D$.

Another approach to strong duality is found in the work of Tyndall [2,4], Levinson [3], and Grinold [5]. These authors impose inequality restrictions on the matrices $B(t)$, $K(t,s)$ and the vectors $a(t)$, $c(t)$ which must be satisfied for each $t$ and $s$. These restrictions are considerably stronger than requiring primal and dual feasibility. They then discretize the time variable to approximate the infinite dimensional programs by finite dimensional programs. The latter are feasible from the inequality assumptions and, under some additional regularity assumptions, they show that the optimal solutions of the finite dimensional programs converge to optimal solutions of the infinite dimensional programs as the time step goes to 0.

5. THE SIMPLEX METHOD

What makes finite dimensional linear programming important is the existence of an efficient algorithm, the simplex method, for computing solutions. The rudiments of a similar algorithm exist in infinite dimensions but one could hardly call it well defined at present. Following Lehman [8] we illustrate the algorithm using the steelmaking example of Section 2 using for convenience the operational calculus notation. Drews, Hartberger and Segers [9] contains similar examples.

We introduce the matrix $P$ whose entries are distributions

$$
P = \begin{pmatrix}
-1_+ & 0_+ & 0 & 0 \\
0 & -1_+ & 0 & 0 \\
0 & 0 & -1_+ & 0 \\
0 & 0 & 0 & -1_+
\end{pmatrix}
$$

where $\delta$ is the Dirac delta function and $1_+$ is the Heavyside function,

$$
1_+(t) = \begin{cases}
0 & t < 0 \\
1 & t \geq 0
\end{cases}
$$

We use $\ast$ to denote the convolution product of (generalized) functions with support in $[0,\infty)$

$$
f \ast x_1(t) = \int_0^\infty f(t-s)x_1(s) \, ds.
$$

In particular

$$
\delta \ast x_1(t) = \int_0^\infty \delta(t-s)x_1(s) \, ds = x_1(t).
$$
The resulting solution is

\[ w \circ f(t) = \int_0^\infty w_1(s)f(t-s) \, ds. \]

(5.6)

which is feasible since

To test the optimality we need

If this is feasible

(5.7)

Define \( \pi(t) \) by

(5.8)

If \( \pi(t) \leq 0 \) then \( w \) is a feasible basis. To

where \( F_1 \) is the

(5.9)

If \( x_1 \geq 0 \) the first

the basis. Our new

As before we solve

(5.10)
The resulting solution is

\[ z(t) = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ c_2 \end{pmatrix} \]

which is feasible since \( c_1 \geq 0 \).

To test the optimality of (5.6) we determine a \( w(t) \) using complementary slackness, i.e.,

\[ \hat{a}(t) - w \circ \hat{P}(t) = 0. \]

If this is feasible then (5.5) is optimal,

\[ w(t) = \hat{a} \circ \hat{P}^{-1}(t) \]
\[ w(t) = (0,0). \]

Define \( \eta(t) \) by

\[ \eta(t) = a(t) - w \circ P(t) \]
\[ \eta(t) = (1, -\omega, 0, 0). \]

If \( \eta(t) \leq 0 \) then \( w \) is dual feasible. If one component of \( \eta(t) \) is positive at some time (as is \( \eta_1(t) \)) we wish to introduce the corresponding component of \( z(t) \) into the basis. To see which variable leaves we look at the equation

\[ \hat{P} \circ \hat{z}(t) = c - z_1 \circ P_1(t) \]

where \( P_1 \) is the first column of \( P \). Multiplying by \( \hat{P}^{-1} \) yields the inequality

\[ \hat{z}(t) = \hat{P}^{-1} \circ c(t) - z_1 \circ \hat{P}^{-1} \circ P_1(t) \geq 0 \]

\[ \begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - z_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \geq 0 \]

If \( z_1 \geq 0 \) the first inequality is always satisfied so it is \( z_4 \) which is dropped from the basis. Our new basis is

\[ z(t) = \begin{pmatrix} z_1(t) \\ z_3(t) \end{pmatrix}. \]

As before we solve (5.5) to obtain the new solution

\[ z(t) = \begin{pmatrix} c_2 \\ 0 \\ c_1 + c_2t \\ 0 \end{pmatrix} \]
We check optimality by computing \( w(t) \) and \( \pi(t) \) from (5.7) and (5.8)

\[
\begin{align*}
w(t) &= (0, 1) \\
\pi(t) &= (0, (T-t)\alpha, 0, -1).
\end{align*}
\]

If \( T \leq \alpha \) then \( \pi(t) \leq 0 \) on \([0, T]\) so \( w(t) \) is dual feasible and (5.10) is the optimal solution.

If \( T > \alpha \) then we wish to introduce \( z_2(t) \) into the basis on the interval \([0, T-\alpha]\) where \( \pi_2(t) \geq 0 \). The new basis changes with time, it is \( z_2 \) and either \( z_1 \) or \( z_3 \) from \([0, T-\alpha]\) and is \( z_1 \) and \( z_3 \) from \([T-\alpha, T]\). We decide which of \( z_1 \) or \( z_3 \) to drop on the first interval by the analog of (5.9)

\[
\begin{align*}
\begin{pmatrix} z_1(t) \\ z_3(t) \end{pmatrix} &= \begin{pmatrix} c_2 \\ c_1 + c_2 \end{pmatrix} - z_2 \begin{pmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \end{pmatrix} + p_2 \geq 0 \\
&= \begin{pmatrix} c_2 \\ c_1 + c_2 \end{pmatrix} - z_2 \begin{pmatrix} 1 \\ \alpha_1 + \alpha_2 \end{pmatrix} \geq 0.
\end{align*}
\]

The first inequality is satisfied for all \( z_2(t) \geq 0 \) so we drop \( z_3(t) \). This turns out to be the optimal solution and involves a delta function.

We have chosen a very simple example to illustrate the method, we refer the reader to [8] and [9] for much more difficult ones.

One unpleasant thing that can happen is that the basis solution could involve delta functions or derivatives of delta functions. Of course the latter are not nonnegative, hence not feasible, but the former are. Another difficulty of this method is knowing over which interval to introduce a new basis variable. One choice is the interval where \( \pi_i(t) \) is positive, but this is not always the right one. Also we should mention that it is possible for a variable to drop out of the basis at one time only to be replaced by another at a different time [9].

These difficulties notwithstanding, this loosely-defined algorithm judiciously employed, does seem to work. It would be immensely useful if it could be rigorously defined and if some kind of convergence established.

6. PIECEWISE BILINEAR CONTROL

In this section we discuss the relationship between autonomous Continuous Linear Programs and Variable Structure Systems. Consider the program of maximizing

\[
(6.1) \quad \int_0^T x(t) \, dt
\]

subject to the constraints

\[
(6.2) \quad z(t) \geq 0
\]

and

\[
(6.3) \quad \text{Such a problem is singular. Hencefor...}
\]

From the above state variable \( x = \maximize z_3(t) \) when

\[
(6.4) \quad \text{and}
\]

\[
(6.5) \quad \text{Differentiating we...}
\]

\[
(6.6) \quad \text{and}
\]

\[
(6.7) \quad \text{Let } \beta = \{ j_1, \ldots \text{ is the } j_i \text{th column matrix whose } j_i \text{...}
\]

\[
(6.8) \quad \text{Let } b_1, \ldots, b_j
\]

It can be shown that

\[
(6.9) \quad \text{where}
\]

\[
(6.10) \quad \text{Combining (6...}
\]
(6.10)

Combining (6.6), (6.7), (6.8), and (6.9) we obtain

\[ x_0 = a^T \beta + \sum_{i=1}^{n} \beta_i y_i \]

where

\[ \Sigma \beta_i y_i \geq 0, \quad \beta_i \geq 0, \quad y_i \geq 0. \]

Differentiating we obtain

\[ x_0 = \frac{\partial F}{\partial x} \]

and

\[ x_0 = \frac{\partial F}{\partial x} \]

Let \( \beta = \{ \beta_1, \ldots, \beta_n \} \) and let \( y_i \) be the non-negative elements of set \( \{ \ldots, y_i \} \). If the matrix \( B \) is non-singular by assumption, let \( \beta_i \) be the non-negative elements of set \( \{ \ldots, \beta_i \} \) and the other rows are zero. We refer to \( \beta_i \) as a basis in \( \{ \ldots, \beta_i \} \). From (6.3) and (6.5) a basis \( \beta_i \) is feasible at \( x \) if

\[ \beta_i \geq 0. \]

It can be shown that if \( \beta \) and \( \beta_i \) both have elements in \( \{ \ldots, \beta_i \} \) then there exists \( v, u \) such that

\[ v_n = -u_n, \quad v_k = u_k, \quad k \neq n. \]

Thus

\[ x_0 = c + \beta y, \]

and

\[ x_0 = c + \beta y \]

such that

\[ x_0 = c + \beta y \]

which is the optimal solution.

From the above we define an optimal control problem of the form

\[ \max \{ x_0 \} \]

where the state variable \( x = (x_0, x_1) \) is a scalar and \( x_1 \) is an old vector. We wish to

\[ x_0 = \int_0^\infty f(x(t)) dt. \]
\[
\dot{x}_1 = \sum_{j} K_{j}^T x_1 + \sum_{j} y_{j}^T K_{j}
\]

This is a piecewise bilinear system in the following sense. Let \( C_\beta \) be the convex cone defined by
\[
C_\beta = \{ z_1 : b_\beta z_1 > 0 \}.
\]

Consider the class of sets with nonempty interior formed by intersections and complements of the \( C_\beta \)'s, called the minimal elements of this class \( \delta_1, \ldots, \delta_n \). They are atoms of the lattice generated by the \( C_\beta \)'s. In general \( \delta_j \) is not a cone but it is closed under multiplication by positive scalars. When \( x_1 \) is restricted to a particular \( \delta_j \), the collection of feasible bases does not change, so on \( \delta_j \) the dynamics (6.10) and (6.11) is bilinear in the usual sense.

This representation of a linear program as an optimal control problem is quite natural. Suppose \( z(t) \), an integrable function, is primal optimal and \( x(t) \) and \( u(t) \), \( v(t) \) are defined by (6.4), (6.5) and (6.8). By the Pontryagin Maximum Principle there exists a dual variable \( \lambda(t) = (\lambda_0(t), \lambda_1(t)) \) which is closely related to the dual of the linear program. If \( w(t) \) is an integrable function which is dual optimal and \( x(t) \in \cup \delta_j \) for almost all \( t \in [0,T] \) then
\[
\lambda_1(t) = \int_t^T w(s) \, ds.
\]

To see this, we define the Hamiltonian
\[
H(\lambda, x, u, v) = \lambda_0 \dot{x}_0 + \lambda_1 \dot{x}_1
\]
\[
= (\lambda_0^T + \lambda_1^T) x + (\lambda_0 u + \lambda_1 v)
\]
\[
= (\lambda_0^T + \lambda_1^T) \left( \sum_{j} b_\beta^T x_1 + \sum_{j} y_{j}^T b_j \right).
\]

The Maximum Principle states that if \( u(t) \), \( v(t) \) and \( x(t) \) are optimal, there exists a \( \lambda(t) = (\lambda_0(t), \lambda_1(t)) \) satisfying the adjoint differential equations
\[
\begin{align*}
\dot{\lambda}_0 &= -\frac{\partial H}{\partial x_0} = 0 \\
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = (\lambda_0^T + \lambda_1^T) \sum b_\beta^T
\end{align*}
\]
the transversality conditions,
\[
\lambda_0(T) \geq 0 , \quad \lambda_1(T) = 0
\]
and for any admissible controls \( u, v \)

\[
(6.17) \quad \text{H}
\]

On the other hand, if
\[
(6.18) \quad \text{H}
\]

If \( x(t) \in \delta_1 \) and \( u_\beta(t) \) or equivalently
\[
(6.19) \quad \text{H}
\]

We multiply by \( u_\beta(t) \)

Define \( \lambda_0(t) = 1 \) conditions and (6.18), differential equation

(6.20) Given any admissible

then

so

On the other hand, if
On the other hand, by complementary slackness conditions and (6.16), we have

\[ \sum_{a \in A} \bar{y}_a \leq 1 \]

so that

\[ \sum_{a \in A} \bar{y}_a \leq 1 \]

Given any admissible \( \bar{y}, \) define

\[ f_{1}^T \bar{y} \leq 1 \]

or equivalently

\[ f_{1}^T \bar{y} = 1 \]

We multiply by \( y_0 \) and sum over \( \beta \) to obtain

\[ y_0 \sum_{a \in A} \bar{y}_a \leq 1 \]

If \( x(\cdot) \in \mathcal{C} \) and \( p_0(c) > 0, \) then \( x(c) > 0 \) for all \( c, \) so

\[ y_0 \sum_{a \in A} \bar{y}_a = \sum_{a \in A} \int_{0}^{1} v(a) \, \text{d}x(a) \]

which is dual optimal and complementary slackness holds, so

\[ x(\cdot) \in \mathcal{C} \]

On the other hand, if \( v(\cdot) \) is dual optimal then complementary slackness holds for almost all \( c, \) so

\[ x(\cdot) \in \mathcal{C} \]

Finally, we have

\[ v(a) = a + \int_{0}^{1} v(a) \, \text{d}x(a) \]

and

\[ v(a) = a + \int_{0}^{1} v(a) \, \text{d}x(a) \]

which implies that for almost all \( c, \) \( x(c) > 0. \)
\[ \nu(t)\xi' = \nu(t)\beta_z(t) = (a + \int_t^T \nu(s) \, ds \, K)z(t) \]

so (6.17) is satisfied.

REFERENCES


ABSTRACT

The purpose of this paper is to show that management of pest populations is similar to the management of pest populations in the environment to support the desired economic and social practices, for example, a short range solution to the problem of pest management. This solution involves the use of ecological principles and the design of new technologies. From the point of view of an ecological system, the problem of pest management is similar to the problem of managing populations in a closed ecosystem.

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