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MINIMUM COVARIANCE, MINIMAX AND MINIMUM ENERGY LINEAR ESTIMATORS

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Abstract: The estimators which minimize the error covariance for the filtering, prediction and smoothing of linear plants with Gaussian initial conditions and noises are well-known. We show that these same estimators arise when one seeks to minimize the maximum error assuming that initial conditions and noises are bounded in norm in an appropriate Hilbert space (minimax estimator). They also arise when one seeks the trajectory of least energy necessary to produce the given observations (minimum energy estimate).

1. Introduction. Consider a linear plant with Gaussian initial condition, driven by Gaussian white noise and observed with additive Gaussian white noise. The problem of optimally estimating the state at time t, given observations up to time τ, is called filtering if $t = \tau$, prediction if $t > \tau$ and smoothing if $t < \tau$. A complete treatment of these problems can be found in [1].

In this paper, which is an extension of [2], we give two alternate characterizations of the minimum covariance filter, smoother and predictor for the linear Gaussian model. These characterizations employ the same linear model but they are nonstochastic, i.e., they do not assume that the unknown initial condition, driving noise and observation noise are stochastic. Instead they assume that these uncertainties lie in a Hilbert space, the norm of which measures the energy of the uncertainties. The norm is related to the covariances of the Gaussian model.

In minimax estimation we assume that the uncertainties are bounded in norm and we seek the estimate of the state which minimizes the maximum possible error given the observations.

The minimum energy estimate assumes that the state of the system is that which is achieved by the uncertainties of least energy necessary to produce the observations.

Both of the above estimators are identical with the minimum covariance estimate. This indicates the robustness of such estimators, and provides an alternative way of looking at the covariances. The latter is particularly desirable because frequently these covariances must be guesstimated when designing an estimator. Finally it is hoped that these the alternative characterizations of linear estimators might lead to computationally feasible nonlinear estimators.

2. The Minimum Covariance, Minimax, and Minimum Energy Estimators. Throughout we consider the time-varying linear system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \\ z(t) &= C(t)x(t) + v(t) \end{aligned} \tag{2.1}$$

where the state $x(t)$ is $n \times 1$, the driving noise $u(t)$ is $\ell \times 1$, the observation $z(t)$ and observation noise $v(t)$ are $m \times 1$. The matrices $A(t)$, $B(t)$ and $C(t)$ are $n \times n$, $n \times \ell$ and $m \times n$ respectively. We assume that (2.1) is a completely controllable system.

A. In minimum covariance estimation we assume that the initial condition x_0 is a Gaussian random vector of mean 0 and covariance

$$E(x_0 x_0') = P_0$$

The driving and observation noises are independent of each other and x_0. They are white Gaussian with zero mean and covariances

$$\begin{aligned} E(u(t)u'(s)) &= Q(t)\delta(t-s) \\ E(v(t)v'(s)) &= R(t)\delta(t-s) \end{aligned}$$

$R(t)$ is assumed to be positive definite but $Q(t)$ and P_0 need only be nonnegative definite. The estimation problem is to find for $t, \tau \in [0,T]$ the estimate $\hat{x}(t|\tau)$ based on the observations $z(s)$, $0 \leq s \leq \tau$ which minimizes the conditional covariance of the error $\tilde{x}(t|\tau) = x(t) - \hat{x}(t|\tau)$, i.e., $\hat{x}(t|\tau)$ minimizes

$$E(b\tilde{x}(t|\tau)\tilde{x}'(t|\tau)b' \,|z(s),\ 0 \leq s \leq \tau)$$

for any $1 \times n$ vector b. Standard statistical results imply that the minimum covariance estimate is the conditional mean

$$\hat{x}(t|\tau) = E(x(t)|z(s),\ 0 \leq s \leq \tau)$$

but one is interested in an efficient method of computing this from the observations.

B. In the minimax model we put a nonstochastic interpretation on the uncertainties x_0, $u(\cdot)$ and $v(\cdot)$. We assume $(x_0, u(\cdot), v(\cdot))$ is an element of a Hilbert space $\mathcal{H}$ and is bounded in norm. For convenience we take the bound to be 1; any bound results in the same estimate although not the same error. The norm is given by

$$\|x_0, u(\cdot), v(\cdot)\|^2 = x_0' P_0^{-1} x_0 + \int_0^T u'(s)Q^{-1}(s)u(s) + v'(s)R^{-1}(s)v(s)\, ds. \tag{2.2}$$

Since P_0 and $Q(t)$ are not necessarily invertible we adopt the following convention. If x_0 is the range of P_0, $x_0 = P_0 y$ then $x_0' P_0^{-1} x_0 = y' x_0$ and if x_0 is not in the range of P_0 then $x_0' P_0^{-1} x_0 = \infty$. This definition is independent of the choice of y since the null space of P_0 is orthogonal to the range of P_0. We adopt a similar convention for $u'(s)Q^{-1}(s)u(s)$. We define $\mathcal{H}$ as the space of triples $(x_0,u(\cdot),v(\cdot))$ satisfying $\|x_0,u(\cdot),v(\cdot)\| < \infty$.

The minimax estimation problem is to find for t, $\tau \in [0,T]$ the estimate $\hat{x}(t,\tau)$ based on the observations $z(s)$, $0 \le s \le \tau$ which minimizes the maximum of any linear functional of the error as $(x_0,u(\cdot),v(\cdot))$ ranges over those triples of norm less than or equal to 1 which give rise to the observations, $z(s)$, $0 \le s \le \tau$. In other words the minimax estimator $\hat{x}(t|\tau)$ minimizes the

$$\max \{b\tilde{x}(t|\tau) : \|x_0,u(\ ),v(\ )\| \le 1 \text{ and produce } z(s),\ 0 \le s \le \tau\}$$

for any $1 \times n$ vector b.

If we fix the observations $z(s)$, $0 \le s \le \tau$, and require that $(x_0,u(\cdot),v(\cdot))$ produce these observations and be of norm ≤ 1 then the set of all possible $x(t)$ is convex because of the linear structure of the model. The minimax estimate is the centroid of this convex set.

The minimax estimator employs a worst case design philosophy and has a game-theoretic flavor. We assume that our opponent, Nature, chooses the uncertainties $(x_0,u(\cdot),v(\cdot))$ in order to hide the true state $x(t)$. Nature is restricted in the total amount of energy (as measured by $\|x_0,u(\cdot),v(\cdot)\|^2$) that she can use. We seek the estimate which minimizes our maximum loss as measured by any linear functional of the error.

C. The minimum energy estimate is characterized in the following fashion. Among all disturbance triples which give rise to the observations $z(s)$, $0 \le s \le \tau$, find the triple of minimum energy. This triple gives rise to a trajectory and the minimum energy estimate $\hat{x}(t|\tau)$ is defined to be the state of that trajectory at time t.

This approach is reminiscent of the variational characterization of certain physical laws, Nature generates the given observations in the most economical way possible and hence the estimate of the state at time t is the state of the minimum energy trajectory at time t.

3. The Equivalence of the Estimators. We begin by showing the minimum energy estimate is equivalent to the minimum covariance estimate. Let $P(t)$ be the solution of the matrix Riccati differential equation

$$\dot{P} = AP + PA' + BQB' - PC'R^{-1}CP$$

(3.1)

$$P(0) = P_0 .$$

The interpretation of P(t) it is well known; it is the conditional error covariance of the minimum covariance filter of Kalman and Bucy,

$$E(b\tilde{x}(t|t)\tilde{x}'(t|t)b' \,|\, z(s),\ 0 \le s \le t) = b\,P(t)\,b'$$

The minimum covariance filter $\hat{x}(t|t)$ satisfies

$$\text{(3.3)} \qquad \frac{d}{dt}\hat{x}(t|t) = A(t)\hat{x}(t|t) + F(t)(z(t)-C(t)\hat{x}(t|t))$$
$$\hat{x}(0|0) = 0$$

where the feedback gain is given by

$$\text{(3.3)} \qquad F(t) = P(t)C'(t)R^{-1}(t).$$

The minimum covariance predictor $\hat{x}(t|\tau)$, $t > \tau$, of Kalman and Bucy is simply the forward extrapolation of $\hat{x}(\tau|\tau)$ assuming no driving noise

$$\text{(3.4)} \qquad \frac{d}{dt}\hat{x}(t|\tau) = A(t)\hat{x}(t|\tau).$$

Rauch, Tung and Striebel have shown that the minimum covariance smoother $\hat{x}(t|\tau)$, $t < \tau$, can be found by integrating backwards from $t = \tau$ the differential equation

$$\text{(3.5)} \qquad \frac{d}{dt}\hat{x}(t|\tau) = A(t)x(t|\tau) + B(t)Q(t)B'(t)P^{-1}(t)(\hat{x}(t|\tau) - \hat{x}(t|t)).$$

Given the observations $z(s)$, $0 \le s \le \tau$, let $(\bar{x}_0, \bar{u}(\cdot|\tau)\ \bar{v}(\cdot|\tau))$ be the triple of minimum energy giving rise to these observations. Let $\bar{x}(\cdot|\tau)$ be the corresponding state trajectory then the minimum energy estimate at time t is $\bar{x}(t|\tau)$.

If $t > \tau$ neither $u(t)$ nor $v(t)$ affect the observations $z(s)$, $0 \le s \le \tau$ so clearly $\bar{u}(t|\tau) = 0$ and $\bar{v}(t|\tau) = 0$. Hence we seek to minimize

$$\text{(3.6)} \qquad x_0'P_0^{-1}x_0 + \int_0^\tau u'(s)Q^{-1}(s)u(s) + v'(s)R^{-1}(s)v(s)\,ds$$

subject to (2.1).

Under the controllability assumption, P(t) is in invertible for all $t > 0$, and

$$\text{(3.7)} \qquad \frac{d}{dt}P^{-1}(t) = -A'P^{-1} - P^{-1}A + C'R^{-1}C - P^{-1}BQB'P^{-1}.$$

Let $\xi(t)$ by a $1 \times n$ vector satisfying

$$\text{(3.8)} \qquad \dot{\xi} = -\xi A - z'R^{-1}C - \xi BQB'P^{-1}$$
$$\xi(0) = 0$$

and let $\varphi(t)$ be a scalar satisfying

$$\dot{\varphi} = z' R^{-1} z - \xi BQB' \xi'$$

(3.9)

$$\varphi(0) = 0$$

If we add the zero quantity

$$(x' P^{-1} x + 2\xi x + \varphi)\Big]_0^\tau - \int_0^\tau \frac{d}{ds} (x' P^{-1} x + 2\xi x + \varphi)\, ds = 0$$

to (3.6) we obtain

$$x'(\tau)P^{-1}(\tau)x(\tau) + 2\xi(\tau)x(\tau) + \varphi(\tau)$$
$$+ \int_0^\tau |Q^{-1/2}(s)u(s) - Q^{1/2'}(s)B'(s)P^{-1}(s)x(s) - Q^{1/2'}(s)B'(s)\xi'(s)|\, ds$$

where $|\cdot|$ is the standard Eculidean norm and $Q = Q^{1/2'} Q^{1/2}$. Clearly $\bar{x}(\tau|\tau)$ is the argument which minimizes

$$x' P^{-1}(\tau)x + 2\xi(\tau)x + \varphi(\tau)$$

i.e.

(3.10) $$\bar{x}(\tau|\tau) = -P(\tau)\xi'(\tau).$$

Furthermore for $t \leq \tau$,

(3.11) $$\bar{u}(t|\tau) = Q(t)B'(t)(P^{-1}(t)\bar{x}(t|\tau) + \xi'(t))$$

and for $t > \tau$

(3.12) $$\bar{u}(t|\tau) = 0.$$

If we differentiate the minimum energy filter (3.10) using (3.7) and (3.8) we see that it satisfies the same differential equation and initial conditions (3.3) as the Kalman-Bucy filter hence they are the same.

Using (2.1), (3.10), (3.11) and (3.12) we see that for $t > \tau$

$$\frac{d}{dt} \bar{x}(t|\tau) = A(t)x(t|\tau)$$

and for $t < \tau$

$$\frac{d}{dt} x(t|\tau) = A(t)\bar{x}(t|\tau) +$$
$$B(t)Q(t)B'(t)P^{-1}(t)(\bar{x}(t|\tau) - \bar{x}(t|t)).$$

These agree with (3.4) and (3.5) therefore the miniman energy estimate $\bar{x}(t|\tau)$ equals the Gaussian estimate $\hat{x}(t|\tau)$ for all $t,\ \tau \in [0,T]$.

Next we show that minimax and minimum energy estimates are equivalent. Let $(x_0, u(\cdot), v(\cdot))$ be any triple giving rise to the observations $z(s)$, $0 \leq s \leq \tau$ and let $(\bar{x}_0, \bar{u}(\cdot), \bar{v}(\cdot))$ be the minimum energy triple for the same observations.

Because it is the minimum energy triple $(\bar{x}_0, \bar{u}(\cdot), \bar{v}(\cdot))$ is orthogonal to any triple $(x_0-\bar{x}_0, u(\cdot)-\bar{u}(\cdot), v(\cdot)-\bar{v}(\cdot))$ giving rise to zero observations on $[0,\tau]$ with respect to the inner product corresponding to (2.2), i.e.

$$\bar{x}_0 P_0^{-1}(x_0-\bar{x}_0) + \int_0^T \bar{u}'(s)Q^{-1}(s)(u(s)-\bar{u}(s)) + \bar{v}'(s)R^{-1}(s)(v(s)-\bar{v}(s))\,ds = 0.$$

If not for some small $\epsilon \neq 0$ the triple

$$(\bar{x}_0, \bar{u}(\cdot), \bar{v}(\cdot)) + \epsilon(x_0-\bar{x}_0, u(\cdot)-\bar{u}(\cdot), v(\cdot)-\bar{v}(\cdot)) \tag{3.13}$$

gives rise to the same observations but is of less energy.

Therefore the norm of (3.13) depends only on $|\epsilon|$ and not on the sign of ϵ. Henceforth we assume $|\epsilon|$ is sufficiently small so that the norm of (3.13) is less than or equal to 1.

Let $x_\epsilon(t)$ be the solution of (2.1) for the triple (3.13), then for $\epsilon = 0$ we have the minimum energy estimate $\bar{x}(t|\tau)$. The linearity of (2.1) implies that

$$x_\epsilon(t) - \bar{x}(t|\tau) = -(x_{-\epsilon}(t) - x(t|\tau)),$$

i.e., the errors are symmetrically distributed around $\bar{x}(t|\tau)$. This shows that $\bar{x}(t|\tau)$ is the centroid of the set of all possible states reachable at time t by a triple of norm less than or equal to 1 which generates the observations $z(s)$, $0 \leq s \leq \tau$, i.e., $\bar{x}(t|\tau)$ is the minimax estimate.

References

[1] A. Gelb, ed., Applied Optimal Estimation, M.I.T. Press, Cambridge, 1974.

[2] A. J. Krener, The Kalman-Bucy filter: an old answer to some new questions in linear filtering, 1978.