A. J. Krener
Department of Mathematics
University of California
Davis, California 95616

Abstract

The assumption of causality for one-dimensional linear systems is dropped. The basic properties of the resulting systems are discussed.

1. Examples. Consider a stiff rod measured by a space like parameter which, in deference to standard control notation, we represent by the letter t. Let $y(t)$ be the deflection of the rod at the point t and suppose at the ends $t = 0, T$ of the rod the deflection is fixed;

$$y(0) = v_1 , y(T) = v_2. \tag{1.1}$$

Let E denote the modulus of elasticity and I the moment of inertia of the cross section of the rod. These may be constant or vary with t. Finally let $u(t)$ denote the bending moment at point t, then we have the standard differential equation for $y(t)$,

$$-EI\ddot{y} = u \tag{1.2}$$

subject to the boundary condition (1.1). We cast these in a form more familiar to system theorist by letting $y = x_1$, $\dot{y} = x_2$ then

$$\dot{x} = Ax + Bu \tag{1.3a}$$
$$y = Cx \tag{1.3b}$$
$$v = V^0x(0) + V^Tx(T) \tag{1.3c}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ -1/EI \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$C = (1 \quad 0) \qquad V^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad V^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Notice that if V^0 is the identity matrix, V^T the zero matrix and $v = x(0)$ then (1.3) is a standard linear system. In general any system of the form (1.3) where x, u, and y are n, ℓ and m vectors and A, B, C, V^0, V^T are arbitrary matrices of the appropriate dimensions possibly time-varying is called a boundary value linear system. As is standard, $x(t)$ is called the state, $u(t)$ the control and $y(t)$ the observable. We reserve the term input for the pair $(v,u(\cdot))$. Notice the acausal character of such systems, in general the state $x(t)$ and observable $y(t)$ depend on values of the control $u(s)$ for all $s \in [0,T]$ not just for $s \in [0,t]$.

Boundary value systems can arise in linear estimation. Suppose we have a noisy signal $z(t)$ generated by the model

$$\dot{x} = Ax + Bu_1 \tag{1.4a}$$
$$z = Cx + u_2 \tag{1.4b}$$
$$x(0) = x^0 \tag{1.4c}$$

where $u_1(\cdot)$ and $u_2(\cdot)$ are white Gaussian noises and x^0 is a Gaussian initial condition, all independent, of zero mean and with specified covariances. From our knowledge of $z(\cdot)$ we seek to estimate $x(t)$. If we are filtering, i.e., using only part information $z(s)$ for $s \leq t$ then the situation is causal but if we are smoothing, i.e., using information $z(s)$ for all $0 \leq s \leq T$ then it is not.

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Moreover in many situations it is incorrect to assume that only information about $x(0)$ is available through the distribution of x^0. Frequently information about both $x(0)$ and $x(T)$ is available, so it might be more accurate to assume the $2n$ vector $(x(0),x(T))$ is Gausian with zero mean and covariance P. Now suppose P is not invertible, then this implies that certain linear functionals of the $2n$ vector $x(0)$ and $x(T)$ are known to be zero surely. Or in other words (1.3c) holds with $v = 0$.

As is discussed in [1], systems such as (1.3) can occur in delay-time and distributed parameter systems. In addition there is a growing interest in multidimensional system theory to be used in the processing of images and other multidimensional data. In such systems the parameters are typically spatial and there is no inherent ordering (or partial ordering) to distinguish the past from the future. Hence such systems are inherently acausal. Before extensive study of them can be undertaken, one-dimensional acausal systems must be fully understood.

Throughout this paper we restrict the continuous time system (1.3) but most of the results carry over immediately to discrete time systems.

2. Input-Output Map. Throughout we assume that (1.3) is well-posed, i.e., there exists only the trivial solution, $x(t) = 0$, of

$$\dot{x} = Ax \tag{2.1a}$$
$$0 = V^0x(0) + V^Tx(T) \tag{2.1b}$$

This a mathematical assumption which guarantees the existence of a unique solution to (1.3) for any input pair $(v,u(\cdot))$. It may not be justified physically but we shall not discuss such situations. Well-posedness is equivalent to the matrix (V^0,V^T) being of rank n and

$$F = V^0 + V^T\Phi(T,0)$$

being invertible. $\Phi(t,s)$ is the fundamental solution of (2.1a),

$$\frac{\partial\Phi}{\partial t}(t,s) = A(t)\Phi(t,s),$$

$\Phi(t,t) = I$. Hence without loss of generality we can assume V^0 and V^T are $n \times n$.

We define the Green's matrix $G(t,s)$

$$G(t,s) = \begin{cases} \Phi(t,0)F^{-1}V^0\Phi(0,s) & \text{if } s < t \\ -\Phi(t,0)F^{-1}V^T\Phi(T,s) & \text{if } t < s \end{cases}$$

then the solution of (1.3) is

$$x(t) = \Phi(t,0)F^{-1}v + \int_0^T F(t,s)B(s)u(s)ds. \tag{2.2}$$

This reduces to the standard variation of constant formula for initial value systems where $V^0 = I$, $V^T = 0$.

We can define a dual system to (1.3), we choose $n \times n$ matrices W^0 and W^T such that the matrix

$$\begin{pmatrix} V^0 & V^T \\ W^0 & W^T \end{pmatrix}$$

is invertible.

We define the output of the system as the pair $(w,y(\cdot))$ where w is defined by

$$w = W^0x(0) + W^Tx(T) \tag{2.3}$$

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Define the input-output map Σ of (1.3) and (2.3) as

$$\Sigma : \mathbb{R}^{n\times 1} \times L_2^{\ell\times 1}[0,T] \to \mathbb{R}^{n\times 1} \times L_2^{m\times 1}[0,T]$$

$$\Sigma : (v,u(\cdot)) \mapsto (w,y(\cdot))$$

where $y(.)$ is defined by (1.3) and w by (2.3). From (2.2) we have

$$y(t) = C(t)\Phi(t,0)F^{-1}v + \int_0^T C(t)G(t,s)B(s)u(s)ds \quad (2.4)$$

Of course Σ has a dual map Σ'

$$\Sigma' : \mathbb{R}^{1\times n} \times L_2^{1\times m}[0,T] \to \mathbb{R}^{1\times n} \times L_2^{1\times \ell}[0,T]$$

$$\Sigma' : (\zeta,\mu(\cdot)) \mapsto (\xi,\nu(\cdot))$$

characterized by the relation

$$\xi v + \int_0^T \nu(t)u(t)dt = \zeta w + \int_0^T \mu(t)y(t)dt.$$

It can be shown [1] that Σ' is the input-output map of the dual system defined by

$$\dot{\lambda} = -\lambda A - \mu C \quad (2.5a)$$

$$\nu = \lambda B \quad (2.5b)$$

$$\zeta = \lambda(0)M^0 + \lambda(T)M^T \quad (2.5c)$$

$$\xi = \lambda(0)N^0 + \lambda(T)N^T \quad (2.5d)$$

for appropriate choices of M^0, M^T, N^0 and N^T. The variables λ, μ and ν are the state, control and observable of the dual system, ζ is the specified boundary conditions (part of the input) analogous to v and ζ is the unspecified boundary conditions (part of the output) analogous to w.

If $V^0 = I$, $V^T = 0$ and we choose $W^0 = 0$, $W^T = I$ then $M^0 = N^T = 0$, $M^T = N^0 = I$ so (2.5) reduces to the familiar dual of an initial value system.

3. Realizability. The observable $y(\cdot)$ is a sum of two terms one depending on the control $u(\cdot)$ and the other on the boundary condition v. We concentrate our attention on the homogeneous boundary situation ($v=0$) since we can always obtain the nonhomogeneous observable by adding the first term of (2.4). In abuse of notation we also denote by Σ the map

$$\Sigma : u(t) \to y(t) = \int_0^T C(t)G(t,s)B(s)u(s)ds \quad (3.1)$$

and define $\Sigma(t,s)$ as the kernel of the map

$$\Sigma(t,s) = C(t)G(t,s)B(s). \quad (3.2)$$

We refer to (3.2) as the weighing pattern or impulse response of the system and say that the system (1.3) is a realization of $\Sigma(t,s)$.

Of course many different systems (1.3) give rise to the same weighing pattern (3.2). One seeks a system of minimal state dimension realizing a given weighing pattern and this is connected with the concepts of controllability and observability.

The system (1.3) and (2.3) is said to be controllable if for any v, w there exists a control $u(\cdot)$ and a solution $x(\cdot)$ of (1.3a) satisfying (1.3c) and (2.3). After a moments reflection one concludes that a well-posed boundary value linear system is controllable iff the corresponding initial value system is, but it is not so clear how to reduce an uncontrollable system to one that is. The details can be found in [1].

The system (1.3) and (2.3) is said to be observable if knowledge of the control $u(\cdot)$ and observable $y(\cdot)$ allows one to determine the boundary conditions v and w. As before a well-posed boundary value system is observable iff the corresponding initial value system is. The details for reducing an unobservable system to one that is are found in [1].

A boundary value system realizing $\Sigma(t,s)$ which is both controllable and observable is said to be a minimal realization, and as is to be expected, such systems are unique up to diffeomorphism of the state space and equivalence of the homogeneous boundary conditions (2.1b), i.e., left-multiplication of V^0 and V^T by an $n \times n$ invertible matrix.

4. Boundary Value Regulators. Returning to our first example of a stiff rod, suppose we wish the deflection $y(t)$ to be small while meeting the boundary condition (1.1) without using excessive bending. We might seek $u(t)$ which minimizes

$$\int_0^T y^2 + u^2\, dt. \quad (4.1)$$

If in addition the rod might be rigidly attached at 0 and T as to keep the stress small we might add to (4.1) a term of the form

$$\dot{y}(0)^2 + \dot{y}(T)^2 \quad (4.2)$$

This is a particular example of a boundary value linear quadratic regulator. In general, we have the system (1.3) and (2.3) and for given v we seek $u(\cdot)$ which minimizes

$$\int_0^T x'Qx + u'Ru\, dt + w'Pw \quad (4.3)$$

where $Q(t)$, $R(t)$ and P are $n \times n$ nonnegative definite matrices.

In our smoothing problem we seek an $n \times m$ kernel $K(t,s)$ which defines the optimal estimate $\hat{x}(t)$ by

$$\hat{x}(t) = \int_0^T K(t,s)z(s)\, ds$$

so as to minimize the mean square of the error $\tilde{x}(t) = x(t) - \hat{x}(t)$. It can be shown using standard techniques that this reduces to a problem generalizing the boundary value linear regulator. If we define an $n \times n$ matrix function $H(t,s)$ by

$$\frac{\partial}{\partial s}H(t,s) = -HA + KC \quad (4.4a)$$

$$H(t,t^-) - H(t,t^+) = I \quad (4.4b)$$

$$H(t,0) = LW^0,\ H(t,T) = -LW^T \quad (4.4c)$$

then for any $1 \times n$ vector b,

$$E(b\tilde{x}(t))^2 = bLPL'b' + \int_0^T bHBQB'Hb' + bKRK'b'\, ds \quad (4.5)$$

where $Q(s)$ and $R(s)$ are the covariances of $u_1(s)$ and $u_2(s)$ and now P denotes the covariance of w given by (2.3). We assume the covariance of v is zero. Except for the jump condition (4.4b), the problem of minimizing (4.5) subject to (4.4) is a boundary value linear quadratic regular where the state is $H'b'$, the control is $K'b'$ and the boundary condition is given in dual form by (4.4c).

How does one solve such a problem? One makes the standard assumptions that $R(t)$ is positive definite for each $t \in [0,T]$ and the system (1.3) and (2.3) is controllable and observable for $C(t)$ satisfying $Q(t) = C'(t)C(t)$. It can be shown that for each v, a unique optimal control $u(t)$ exists and is a linear function of v. The optimal cost is quadratic in v.

To find the optimal controls, one can use the Maximal Principle and solve the resulting two point boundary value problem for n linear independent boundary conditions. Let V be a nonsinular n X n matrix, each column of which is to be interpreted as a boundary condition for (1.3c). Let X(t) and Λ(t) be n X n matrices, each column of X(t) and corresponding row of Λ(t) are an optimal state and costate pair. By the maximal principle they satisfy the Hamiltonian system

$$\dot{X} = AX - BR^{-1}B'\Lambda' \quad (4.6a)$$

$$\dot{\Lambda} = -\Lambda A - X'Q \quad (4.6b)$$

subject to the boundary and transversality conditions

$$V^0X(0) + V^TX(T) = V, \quad (4.7a)$$

$$W^0X(0) + W^TX(T) = W, \quad (4.7b)$$

$$\Lambda(0) = -W'PW^0 + \Xi V^0, \quad (4.7c)$$

$$\Lambda(T) = W'PW^T - \Xi V^T \quad (4.7d)$$

for some matrix Ξ.
The optimal controls corresponding to the columns of X(t) are the columns of U(t) given by

$$U = -R^{-1}B'\Lambda' \quad (4.8)$$

For arbitrary initial condition v the corresponding optimal u(t), x(t) and λ(t) are given by

$$u(t) = U(t)V^{-1}v, \quad (4.9a)$$

$$x(t) = X(t)V^{-1}v, \quad (4.9b)$$

$$\lambda(t) = v'V^{-1'}\Lambda(t). \quad (4.9c)$$

Motivated by the initial value linear quadratic regulator one might seek to solve this problem via a Riccati differential equation. For simplicity let's assume that the boundary condition (1.3c) separates. If we make an appropriate change of coordinates $x = (x_1,x_2)$ and $v = (v_1,v_2)$, (vectors of dimensions k and n-k respectively), and (1.3c) becomes

$$x_1(0) = v_1, \ x_1(T) = v_2. \quad (4.10)$$

We further assume that the boundary cost of (4.3) separates, that is, for some n X n matrices P_0 and P_T, (4.3) becomes

$$\int_0^T x'Qx + u'Ru\,dt + x'(0)P_0x(0) + x'(t)P_Tx(T) \quad (4.11)$$

We seek a solution of the n X n matrix Riccati equation

$$\dot{K} = -KA - A'K - Q + KBR^{-1}B'K \quad (4.12)$$

satisfying the boundary condition

$$K(T) = P_T \quad (4.13)$$

If we add the identically zero quantity

$$\int_0^T \frac{d}{dt}(x'Kx)\,dt - x'Kx]_0^T$$

to (4.11) we obtain

$$\int_0^T \|R^{-1/2'}B'Kx + R^{1/2}u\|^2 dt + x'(0)(K(0) + P_0)x(0) \quad (4.14)$$

Clearly the control which minimize (4.14) is given by

$$u = R^{-1}B'Kx \quad (4.15)$$

For any v_1 we solve the problem of minimizing

$$x'(K(0) + P_0)x$$

subject to the constraint that the first k coordinates of x equal v_1. This gives us x(0), we compute x(t) using (4.15) and (1.3a) and finally we compute x(T) and v_2 from (4.10).

Using K(t) we have found a solution of our original problem satisfying the boundary condition (4.10), and by varying v_1 through k linearly independent values we can find k solutions from the same K(t). To find additional solutions, n linearly independent ones exist, we must compute a new solution of the Riccati equation (4.12) satisfying the new boundary condition

$$K_T = P_T + \Pi$$

where Π is any nonnegative definite matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

Adding Π only changes the optimal cost, it does not affect the optimal controls because the last n-k components of x(T) are fixed by (4.10). In this way, hopefully we obtain k more linearly independent solutions to the original problem. We continue the process until we have n linearly independent over.

5. Feedback Controls. It is well-known that the optimal solution of the initial value linearly quadratic regulator can be expressed in feedback form (4.15), is a similar development possible for boundary value regulators? The answer is no as we shall show in a moment by counterexample. In section 4 we solved such a proble by multiple solutions of the matrix Riccati equation. We found n linearly independent optimal solutions $x^i(t)$, and $u^i(t)$ related by (4.15) but for different solutions $K^i(t)$ of the Riccati equation.

$$u^i = -R^{-1}B'K^ix^i$$

a general solution x(t) and u(t) is a linear combination of the above

$$x(t) = \Sigma\mu_i x^i(t)$$

$$u(t) = \Sigma\mu_i u^i(t)$$

but this does not mean that u(t) can be computed from x(t),

$$u = \Sigma\mu_i u^i = \Sigma\mu_i R^{-1}B'K^ix^i.$$

If X(t), Λ(t) and U(t) are as in section 4 and $X^{-1}(t)$ for each $t \in [0,T]$ then it is not hard to see that $K = \Lambda'X^{-1}$ is a solution of the Riccati equation (4.12) and the optimal controls are given by

$$U = -R^{-1}BKX$$

In other words the problem is solvable in feedback form. It is worth noting that $K = \Lambda'X^{-1}$ need not be positive definite or even symmetric.

In the language of the calculus of variations, the condition that $X^{-1}(t)$ exists for $t \in [0,T]$ is just that there exists no focal points for the problem. But focal points do exist for even the simplest of these problems.

Consider the stiff rod discussed in sections 1 and 4 with the optimal cost determined 4.1 and the boundary condition (1.1). If $T = k\pi/\alpha$ where $\alpha = \sqrt{2}/2$ then the Hamiltonian system (4.6) and (4.7) is easy to solve. For V equal to

$$V = \begin{pmatrix} 1 & 1 \\ (-e^{\pi})^k & (-e^{-\pi})^k \end{pmatrix}$$

$$X(t) = \begin{pmatrix} e^{\alpha t}\cos\alpha t & e^{-\alpha t}\cos\alpha t \\ \alpha e^{\alpha t}(\cos\alpha t - \sin\alpha t) & \alpha e^{-\alpha t}(\sin\alpha t - \cos\alpha t) \end{pmatrix}$$

which is singular at $t = (\ell + 1/2)\pi/\alpha$ for $\ell = 0,\dots,k-1$. Notice there can be an arbitrary large number of focal points. Each focal point corresponds to the zero of a nontrivial optimal solution. Using controllability it can be shown that no nontrivial solution has more than one zero.

Reference

[1] Krener, A. J., Boundary Value Linear Systems, to appear, Asterique.