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ON THE EQUIVALENCE OF CONTROL SYSTEMS AND THE LINEARIZATION OF NONLINEAR SYSTEMS*

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Abstract. Given two control systems where the control enters linearly, a necessary and sufficient condition is derived that these systems be locally diffeomorphic, i.e., that there exist a local diffeomorphism between the state spaces which carries a trajectory of the first system for each control into the trajectory of the second system for the same control. As a corollary we derive necessary and sufficient conditions for a system to be locally diffeomorphic to a linear system.

1. Introduction. Consider the two control systems

$$\begin{aligned} \dot{x} &= a_0(x) + \sum_{i=1}^{k} u_i(t)a_i(x), \\ x(0) &= x^0, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \dot{y} &= b_0(y) + \sum_{i=1}^{k} u_i(t)b_i(y), \\ y(0) &= y^0, \end{aligned} \tag{2}$$

where $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_n)$ are vectors, $a_0(x), \cdots, a_k(x), b_0(x), \cdots, b_k(x)$ are analytic vector-valued functions and $u(t) = (u_1(t), \cdots, u_k(t))$ is a bounded measurable control.

The purpose of this paper is to give necessary and sufficient conditions that these two systems be equivalent, i.e., that there exist a local diffeomorphism from x-space to y-space which takes the solution of (1) for each control into the solution of (2) for the same control. As a corollary we derive necessary and sufficient conditions that there exist a local diffeomorphism which carries a nonlinear system into a linear one.

2. Preliminaries. If $a_i(x), a_j(x)$ are as above we define the Lie bracket $[a_i, a_j](x)$, another analytic vector-valued function, by

$$[a_i, a_j](x) = \frac{\partial a_j}{\partial x}(x)a_i(x) - \frac{\partial a_i}{\partial x}(x)a_j(x),$$

where $(\partial a_j/\partial x)(x)$ is the matrix of partial derivatives at x. Suppose $t, x \mapsto \alpha_i(t)x$ is the family of integral curves of $a_i(x)$, that is, $(d/dt)\alpha_i(t)x = a_i(\alpha_i(t)x)$ and $\alpha_i(0)x = x$. Then for fixed t, the map $x \mapsto \alpha_i(-t)x$ is a diffeomorphism from a neighborhood of $\alpha_i(t)x^0$ onto a neighborhood of x^0 and hence has a tangent map which we denote by $\alpha_i(-t)_*$. The derivative of the vector-valued curve $t \mapsto \alpha_i(-t)_* a_j(\alpha_i(t)x^0)$ at $t = 0$ is $[a_i, a_j](x^0)$ (Bishop and Crittenden [1, p. 17]). Since a_i, a_j are analytic, we obtain the Taylor series expansion $\alpha_i(-t)_* a_j(\alpha_i(t)x^0) = \sum_{h=0}^{\infty} (t^h/h!) ad^h(a_i)a_j(x^0)$, where $ad^0(a_i)a_j(x^0) = a_j(x^0)$ and $ad^h(a_i)a_j(x^0) = [a_i, ad^{h-1}(a_i)a_j](x^0)$.

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Following Haynes and Hermes [2] we define $D^0(A)$ to be a set of functions $\{a_i : i = 0, \cdots, k\}$ and $D^j(A) = D^{j-1}(A) \cup \{[a_i, c] : i = 0, \cdots, k,\ c \in D^{j-1}(A)\}$, for $j \leq 1$. The *completed system* of A is $D(A) = \bigcup_{j \geq 0} D^j(A)$, and we define $D(A)_x = \{c(x) : c \in D(A)\} \subseteq \mathbb{R}^m$. The *rank r of D(A) at x* is just the dimension of the span $D(A)_x$.

THEOREM (Nagano [4]). *Let the completed system of* (1) *have rank r at* x^0. *Then there exists a submanifold M of dimension r through* x^0, *which carries* (1). *That is, if* $u(t)$ *is any bounded measurable control and* $x(t)$ *is the corresponding solution of* (1), *then for some* $\varepsilon > 0$, $x(t) \in M$ *for* $|t| < \varepsilon$.

For generalizations of this result see Krener [3].

3. Equivalent systems.

THEOREM 1. *Consider the systems* (1) *and* (2). *Let M and N be submanifolds which carry* (1) *and* (2) *at* x^0 *and* y^0 *respectively. There exists a linear map* $l : \text{span } D(A)_{x^0} \to \text{span } D(B)_{y^0}$ *such that* $l(a_i(x^0)) = b_i(y^0)$ *for* $i = 0, \cdots, k$ *and*

$$l([a_{i_1}, \cdots [a_{i_{h-1}}, a_{i_h}] \cdots](x^0)) = [b_{i_1}, \cdots [b_{i_{h-1}}, b_{i_h}] \cdots](y^0)$$

for $h \leq 2$ *and* $1 \leqq i_j \leqq k$ *if and only if there exist neighborhoods U and V of* x^0 *and* y^0 *in M and N and an analytic map* $\lambda : U \to V$ *such that* λ *carries* (1) *into* (2). *That is, if* $x(t)$ *and* $y(t)$ *are the solutions of* (1) *and* (2) *for the same control* $u(t)$ *and* $x(t) \in U$ *for* $|t| < \varepsilon$, *then* $y(t) = \lambda(x(t)) \in V$ *for* $|t| < \varepsilon$. *Furthermore l is a linear isomorphism if and only if* λ *is a local diffeomorphism.*

Proof. We start by assuming l exists and constructing λ. Since the theorem is local in nature, we can assume that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, then span $D(A)_{x^0} = \mathbb{R}^m$ and span $D(B)_{y^0} = \mathbb{R}^n$. Let $c_1(x^0), \cdots, c_h(x^0)$ be a maximal linearly independent subset of $D^0(A)_{x^0}$. Let $d_1(y), \cdots, d_h(y)$ be the corresponding elements of $D^0(B)$, that is, if $c_i(x) = a_j(x)$ then $d_i(y) = b_j(y)$. We choose $c_{h+1}(x), \cdots, c_m(x)$ from $D(A)$ so that $c_1(x^0), \cdots, c_m(x^0)$ forms a basis for $\mathbb{R}^m$. Let $d_{h+1}(y), \cdots, d_m(y)$ be the corresponding elements of $D(B)$, that is, if $c_i(x) = [a_{j_1}, \cdots [a_{j_{l-1}}, a_{j_l}] \cdots](x)$, then $d_i(y) = [b_{j_1}, \cdots [b_{j_{l-1}}, b_{j_l}] \cdots](y)$. Let $t, x \mapsto \alpha_i(t)x$, be the family of integral curves of $c_i(x)$ for $i = 1, \cdots, m$. That is $(d/dt)\alpha_i(t)x = c_i(\alpha_i(t)x)$ and $\alpha_i(0)x = x$. Similarly $t, y \mapsto \beta_i(t)y$, is defined by $(d/dt)\beta_i(t)y = d_i(\beta_i(t)y)$ and $\beta_i(0)y = y$, for $i = 1, \cdots, m$. Let $s = (s_1, \cdots, s_m)$ and define maps $g_1 : s \mapsto x$ and $g_2 : s \mapsto y$ by $g_1(s) = \alpha_m(s_m) \cdots \alpha_2(s_2)\alpha_1(s_1)x^0$ and $g_2(s) = \beta_m(s_m) \cdots \beta_2(s_2)\beta_1(s_1)y^0$. Then $(\partial g_1/\partial s_i)(0) = c_i(x^0)$, so g_1 has an inverse $g_1^{-1} : x \mapsto s$ defined for x in some neighborhood U of x^0. Let $\lambda : x \mapsto y$ be defined on U by $\lambda = g_2 \circ g_1^{-1}$.

We must now show that if $x(t)$ and $y(t)$ are the solutions of (1) and (2) respectively for the same control $u(t)$, then $\lambda(x(t)) = y(t)$. Since $\lambda(x(0)) = \lambda(x^0) = y^0 = y(0)$ it suffices to show that $(d/dt)\lambda(x(t)) = (d/dt)y(t)$ or $\lambda_*(\dot{x}(t)) = \dot{y}(t)$, where λ_* is the tangent map to λ at $x(t)$. This is true if $\lambda_*(a_i(x)) = b_i(\lambda(x)), i = 1, \cdots, k$, for all $x \in U$, which in turn would follow if $\lambda_*(c_i(x)) = d_i(\lambda(x))$, $i = 1, \cdots, m$, for all $x \in U$.

To show this we let $x = g_1(s)$, $x^i = g_1(s_1, \cdots, s_i, 0, \cdots, 0)$, for $i = 1, \cdots, m$, $y = \lambda(x) = g_2(s)$ and $y^i = g_2(s_1, \cdots, s_i, 0, \cdots, 0)$, for $i = 1, \cdots, m$. Then $x^m = x$ and for $i = 1, \cdots, m$, the map $\alpha_i(-s_i)(\cdot)$ takes x^i into x^{i-1} and is a local diffeomorphism with tangent at x^i denoted by $\alpha_i(-s_i)_*$. Similarly $y^m = y$, and the map

$\beta_i(s_i)(\cdot)$ takes y^{i-1} into y^i and is a local diffeomorphism with tangent at y^{i-1} denoted by $\beta_i(s_i)_*$.

We now show that $\lambda_* = \beta_m(s_m)_* \cdots \beta_1(s_1)_* \, l \, \alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_*$. Since $\partial g_1(s)/\partial s_i$ forms a basis for $\mathbb{R}^m$, it suffices to show that the right side applied to $\partial g_1(s)/\partial s_i$ yields $\lambda_*(\partial g_1(s)/\partial s_i)$ which equals $\partial g_2(s)/\partial s_i$. But $\partial g_1(s)/\partial s_i = \alpha_m(s_m)_* \cdots \alpha_i(s_i)_* c_i(x^{i-1})$ and $\partial g_2(s)/\partial s_i = \beta_m(s_m)_* \cdots \beta_i(s_i)_* d_i(y^{i-1})$ so

$$\begin{aligned}
&\beta_m(s_m)_* \cdots \beta_1(s_1)_* l \alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_* \frac{\partial g_1(s)}{\partial s_i} \\
&\quad = \beta_m(s_m)_* \cdots \beta_1(s_1)_* l \alpha_1(-s_1)_* \cdots \alpha_{i-1}(-s_{i-1})_* c_i(x^{i-1}) \\
&\quad = \beta_m(s_m)_* \cdots \beta_1(s_1)_* l \sum \frac{(s_1)^{h_1}}{h_1!} ad^{h_1}(c_1)\left(\cdots \sum \frac{(s_{i-1})^{h_{i-1}}}{(h_{i-1})!} ad^{h_{i-1}}(c_{i-1})c_i \cdots\right)(x^0) \\
&\quad = \beta_m(s_m)_* \cdots \beta_1(s_1)_* \sum \frac{(s_1)^{h_1}}{h_1!} ad^{h_1}(d_1)\left(\cdots \sum \frac{(s_{i-1})^{h_{i-1}}}{(h_{i-1})!} ad^{h_{i-1}}(d_{i-1})d_i \cdots\right)(y^0) \\
&\quad = \beta_m(s_m)_* \cdots \beta_i(s_i)_* d_i(y^{i-1}) = \frac{\partial g_2(s)}{\partial s_i}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\lambda_*(c_i(x^m)) &= \beta_m(s_m)_* \cdots \beta_1(s_1)_* l \alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_* c_i(x^m) \\
&= \beta_m(s_m)_* \cdots \beta_1(s_1)_* l \sum \frac{(s_1)^{h_1}}{h_1!} ad^{h_1}(c_1)\left(\cdots \sum \frac{(s_m)^{h_m}}{h_m!} ad^{h_m}(c_m)c_i \cdots\right)(x^0) \\
&= \beta_m(s_m)_* \cdots \beta_1(s_1)_* \sum \frac{(s_1)^{h_1}}{h_1!} ad^{h_1}(d_1)\left(\cdots \sum \frac{(s_m)^{h_m}}{h_m!} ad^{h_m}(d_m)d_i \cdots\right)(y^0) \\
&= d_i(y^m).
\end{aligned}$$

Notice that $\lambda_*(c_i(x^0)) = d_i(y^0)$ so $l = \lambda_*$ at x^0. It follows by the inverse function theorem that if l is a linear isomorphism then λ is a local diffeomorphism.

As for the converse, if λ exists and $\lambda(x(t)) = y(t)$ where $x(t)$ and $y(t)$ are the solutions of (1) and (2) for the same control $u(t)$, then clearly $\lambda_*(a_i(x)) = b_i(\lambda(x))$. It is a standard result of differential geometry (Bishop and Crittenden [1, p. 14]) that if $\lambda_*(c_i(x)) = d_i(\lambda(x))$, $i = 1, 2$, then $\lambda_*([c_1, c_2](x)) = [d_1, d_2](\lambda(x))$, and so $l = \lambda_*$ at x^0 satisfies the required condition. Q.E.D.

Remark. Since $g_1(s)$ covers a neighborhood of x^0 in M, the map λ is uniquely determined in that neighborhood by the condition that it take system (1) into system (2). Furthermore if M is connected and simply connected, then λ can be extended uniquely to a map defined on all M by standard arguments. See Example 3 below.

Example 1. Consider the two systems

$$\dot{x}_1 = u, \qquad \dot{y}_1 = u,$$

$$\dot{x}_2 = u \cdot t, \quad \dot{y}_2 = y_1.$$

Since the right-hand side of the first system depends on t, we introduce a new variable $x_0 = t$.

$$\begin{aligned} \dot{x}_0 &= 1, \\ \dot{x}_1 &= u, \\ \dot{x}_2 &= u \cdot x_0, \end{aligned} \qquad a_0(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad a_1(x) = \begin{pmatrix} 0 \\ 1 \\ x_0 \end{pmatrix}, \quad [a_0, a_1](x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$b_0(y) = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}, \quad b_1(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_0, b_1](y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

and all other brackets are zero.

For initial points $x^0 = (x_0^0, x_1^0, x_2^0)$, $y^0 = (y_1^0, y_2^0)$, let $l: \mathbb{R}^3 \mapsto \mathbb{R}^2$ be given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ y_1^0 & x_0^0 & -1 \end{pmatrix}.$$

The hypotheses of Theorem 1 are satisfied and λ can be constructed as in the theorem. Let α_1, α_2 and α_3 be the families of integral curves of a_0, a_1 and $[a_0, a_1]$ and β_1, β_2 and β_3 be the families of integral curves of b_0, b_1 and $[b_0, b_1]$. Then

$$g_1(s_1, s_2, s_3) = \alpha_3(s_3)\alpha_2(s_2)\alpha_1(s_1)x^0 = (x_0^0 + s_1, x_1^0 + s_2, x_2^0 + (x_0^0 + s_1)s_2 + s_3),$$

$$g_2(s_1, s_2, s_3) = \beta_3(s_3)\beta_2(s_2)\beta_1(s_1)y^0 = (y_1^0 + s_2, y_2^0 + s_1 y_1^0 - s_3),$$

and

$$\lambda(x) = g_2(g_1^{-1}(x)) = (y_1^0 + x_1 - x_1^0, y_2^0 + (x_0 - x_0^0)y_1^0 - (x_2 - x_2^0) + x_0(x_1 - x_1^0)).$$

Notice that $M = \mathbb{R}^3$, $N = \mathbb{R}^2$ and λ is defined for all $x \in \mathbb{R}^3$ and is onto $\mathbb{R}^2$. In fact, if we introduce a time coordinate $y_0 = t$ into the second system, then $N = \mathbb{R}^3$ and λ becomes a diffeomorphism from $\mathbb{R}^3$ onto $\mathbb{R}^3$:

$$\begin{aligned} \lambda(x) = (&y_0^0 + x_0 - x_0^0, y_1^0 + x_1 - x_1^0, y_2^0 + (x_0 - x_0^0)y_1^0 - (x_2 - x_2^0) \\ &+ x_0(x_1 - x_1^0)). \end{aligned}$$

Example 2. Suppose we replace the second system of Example 1 with one similar to that of Haynes and Hermes [2]·

$$\begin{aligned} \dot{y}_0 &= 1, \\ \dot{y}_1 &= u, \\ \dot{y}_2 &= u \cdot y_0 \cdot y_2, \end{aligned} \qquad b_0(y) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b_1(y) = \begin{pmatrix} 0 \\ 1 \\ y_0 \cdot y_2 \end{pmatrix} \quad \text{and} \quad [b_0, b_1](y) = \begin{pmatrix} 0 \\ 0 \\ y_2 \end{pmatrix},$$

and all other brackets are identically zero. The rank of $D(B)$ is 3 except at points where $y_2 = 0$, where it is 2. The system splits $\mathbb{R}^3$ into three disjoint manifolds $N_+ = \{y : y_2 > 0\}$, $N_0 = \{y : y_2 = 0\}$ and $N_- = \{y : y_2 < 0\}$. A trajectory of this system must lie wholly within one of these manifolds.

For initial points x^0 and y^0 we define l by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & (y_0^0 - x_0^0)y_2^0 & y_2^0 \end{pmatrix}$$

and construct λ as before:

$$\lambda(x) = (y_0^0 + x_0 - x_0^0, y_1^0 + x_1 - x_1^0, y_2^0 \exp((y_0^0 - x_0^0)(x_1 - x_1^0) + x_2 - x_2^0)).$$

Notice if $y^0 \in N_+(N_-)$, then λ is a diffeomorphism $\lambda : \mathbb{R}^3 \to N_+(N_-)$. If $y^0 \in N_0$, then $\lambda : \mathbb{R}^3 \to N_0$ is onto.

Example 3. Consider the systems

$$\dot{x}_1 = -ux_2, \qquad \dot{y}_1 = u,$$

$$\dot{x}_2 = ux_1,$$

$$a_1 = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \qquad b_1 = (1),$$

and of course there are no nontrivial brackets. If $x^0 = (1, 0)$ and $y^0 = 0$, then $M = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ and $N = \mathbb{R}$. So $l = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfies Theorem 1 and λ is defined in a neighborhood of $(1, 0)$ on M by $\lambda(x_1, x_2) = \arctan(x_2/x_1)$. It is clear that λ cannot be extended to a map on all of M.

4. The linearization of nonlinear systems. Consider the linear control system

$$\dot{y} = F(t)y(t) + G(t)u(t) + h(t), \tag{3}$$

where F and G are matrices, y and h are vectors and u is the control vector. As before we introduce time as a coordinate, $y_0 = t$. It is well known that there exists a change of the y coordinates which carries (3) into

$$\dot{y} = b_0 + \sum_{i=1}^{k} u_i b_i(y_0), \tag{4}$$

where

$$b_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad b_i = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix}, \quad i = 1, \cdots, k \quad \text{and} \quad y_0^0 = 0,$$

and where * denotes some real-valued function of y_0 alone.

The question we now answer is when does there exist a transformation $\lambda : x \mapsto y$ which carries a nonlinear system (1) into a linear system (4).

THEOREM 2. *Consider the system* (1). *Let* n = *rank of* $D(A)_{x^0}$ *and let* M *be the* n*-dimensional manifold which carries* (1). *There exists a linear system* (4), *a neighborhood* U *of* x^0 *in* M, *a neighborhood* V *of* $y^0 = 0$ *in* $\mathbb{R}^n$ *and a diffeomorphism* $\lambda : U \mapsto V$ *carrying* (1) *into* (4) *if and only if for all* $1 \leqq i, j \leqq k$ *and for all* $h \geqq 0$, $[a_i, ad^h(a_0)a_j](x^0) = 0$.

Proof. Suppose the system (4) and λ exist. Then λ_*, the tangent to λ at 0, is one-to-one and

$$\lambda^*([a_i, ad^h(a_0)a_j](x^0)) = [b_i, ad^h(b_0)b_j](0).$$

Then by induction for $h \geqq 0$,

$$ad^h(b_0)b_j = \begin{pmatrix} 0 & \cdots & 0 \\ * & & \\ \vdots & 0 & \\ * & & \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} \\ 0 \\ \\ \end{pmatrix}\begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix}$$

and

$$[b_i, ad^h(b_0)b_j] = \begin{pmatrix} 0 & \cdots & 0 \\ * & & \\ \vdots & 0 & \\ * & & \end{pmatrix}\begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 \\ * & & \\ \vdots & 0 & \\ * & & \end{pmatrix}\begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and it follows that $[a_i, ad^h(a_0)a_j] = 0$.

On the other hand if $[a_i, ad^h(a_0)a_j](x^0) = 0$, we construct (4) as follows. Let $s, x \to \alpha_0(s)x$ be the family of integral curves of a_0. Define the system (4) by setting

$$b_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad b_j(y_0) = \alpha_0(-y_0)_* a_j(\alpha_0(y_0)x^0), \quad j = 1, \cdots, k.$$

The Taylor series expansion of $b_j(y_0) = \sum_{h=0}^{\infty} ((y_0)^h/h!) ad^h(a_0)a_j(x^0)$ and it follows that

$$ad^h(b_0)b_j(0) = \frac{d^h}{dy_0^h} b_j(0) = ad^h(a_0)a_i(x^0) \quad \text{for } j = 1, \cdots, k \text{ and } h \geqq 0.$$

Also by hypothesis $[a_i, ad^h(a_0)a_j](x^0) = 0$ and we showed above for systems of type (4), $[b_i, ad^h(b_0)b_j](0) = 0$. Therefore the hypotheses of Theorem 1 are satisfied with l = identity map, and so we can construct λ. Q.E.D.

Example 4. Consider the nonlinear system

$$\dot{x}_1 = 1 + u \cdot x_3,$$

$$\dot{x}_2 = x_1^2 x_2 + u,$$

$$\dot{x}_3 = x_3,$$

$$a_0 = \begin{pmatrix} 1 \\ x_1^2 x_2 \\ x_3 \end{pmatrix}, \quad a_1 = \begin{pmatrix} x_3 \\ 1 \\ 0 \end{pmatrix}, \quad [a_0, a_1] = \begin{pmatrix} x_3 \\ -2x_1x_2x_3 - x_1^2 \\ 0 \end{pmatrix}$$

$$\text{and} \quad [a_1[a_0, a_1]] = \begin{pmatrix} 0 \\ -2x_2x_3^2 - 4x_1x_3 \\ 0 \end{pmatrix}.$$

Therefore the system is not linearizable in general. However if $x_3^0 = 0$, then the system is carried by $M = \{x : x_3 = 0\}$ and on this submanifold

$$a_0 = \begin{pmatrix} 1 \\ x_1^2 x_2 \\ 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad [a_0, a_1] = \begin{pmatrix} 0 \\ -x_1^2 \\ 0 \end{pmatrix}, \quad [a_1[a_0, a_1]] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$ad^2(a_0)a_1 = \begin{pmatrix} 0 \\ -2x_1 \\ 0 \end{pmatrix}, \quad [a_1, ad^2(a_0)a_1] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad ad^3(a_0)a_1 = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}.$$

All higher brackets are zero, so the system is linearizable. We do not have to compute λ to describe the equivalent linear system. For example if $x^0 = (0, 0, 0)$ we define

$$b_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b_1(y_0) = \sum_{h=0}^{\infty} \frac{y_0^h}{h!} ad^h(a_0)a_1(x^0) = \begin{pmatrix} 0 \\ 1 - \dfrac{y_0^3}{3} \\ 0 \end{pmatrix}.$$

Since $y_0 = t$ and the y_2-coordinate is superfluous, this becomes

$$\dot{y}_1 = u(1 - t^3/3), \quad y_1(0) = 0.$$

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