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The Complexity of Stochastic Differential Equations

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There has been considerable interest lately in the complexity of solving stochastic differential equations, for example, can they be solved individually for each sample path. In this note we unify what several researchers have indicated, namely that the stochastic complexity depends on the Lie algebra generated by the vector fields multiplying the noises and does not depend on the drift term.

1. STATEMENT OF THE RESULTS

Consider the stochastic differential system

$$dX = f(X)\,dt + g(X)\,dW \tag{1.1a}$$

$$X(0) = X^0 \tag{1.1b}$$

where X is an $n \times 1$ vector valued process, W is standard Wiener process of dimension $m \times 1$, X^0 is $n \times 1$ dimensional random vector independent of W, and f, g are smooth (C^∞) functions of x with values of dimensions $n \times 1$ and $n \times m$ respectively. Nonautonomous systems are included in (1.1) by letting one component of X be equal to t.

The purpose of this note is to show that under certain assumptions the complexity of solving (1.1) introduced by the stochastic nature of W is

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independent of the drift vector field $f(x)$ and depends only on the vector fields $g^1(x),\ldots,g^m(x)$ comprising the columns of $g(x)$. Intuitively this is quite reasonable because the latter are multiplied by dW_i which is "of order $\sqrt{dt}$" while the former is multiplied by dt. An immediate corollary of this fact is that the complexity introduced by the stochastics is the same whether (1.1) is interpreted as an Ito or Stratonovich differential equation, since the transition from one to the other only involves modification of the drift term.

Let us be more precise about what we mean by the complexity of (1.1) introduced by the stochastics. But first what about the complexity of the system without stochastics, i.e., consider the analogous control system

$$\dot{x} = f(x) + g(x)u \tag{1.2a}$$

$$x(0) = x^0 \tag{1.2b}$$

where $x(t)$ and $u(t)$ are sure functions. Here we are interpreting (1.1) in the Stratonovich sense; if (1.1) is an Ito equation then the appropriate deterministic analog is

$$\dot{x} = \tilde{f}(x) + g(x)u \tag{1.3a}$$

$$x(0) = x^0 \tag{1.3b}$$

where the drift is modified by a correction term

$$\tilde{f}(x) = f(x) + \frac{1}{2}\sum_{j=1}^{m} \frac{\partial g^j}{\partial x}(x)\, g^j(x). \tag{1.3c}$$

Intuitively, there are two ways of approaching the complexity of (1.2) (or (1.3)). The first is how hard is it to integrate? For arbitrary $u(t)$ can the solution be written in closed form involving quadratures? The second is how difficult is it to simulate? Suppose we have another system

$$\dot{z} = h(z, v) \tag{1.4a}$$

$$z(0) = z^0 \tag{1.4b}$$

and maps $v = \alpha(u)$, $x = \varphi(z)$ such that for any input $u(t)$, the solution $x(t)$ of (1.2) and the solution $z(t)$ of (1.4) under input $v(t) = \alpha(u(t))$ satisfy $x(t) = \varphi(z(t))$. In this case the second system (1.4) is said to *simulate* the first (1.2) and hence intuitively (1.4) must be at least as complex as (1.2).

It turns out the answer to both of the above questions involves the Lie algebra $\mathscr{F}$ of vector fields generated by $\{f, g^1,\ldots, g^m\}$. The solutions to (1.2) can be expressed in quadratures iff the algebra $\mathscr{F}$ is solvable. If $\mathscr{F}$ is nilpotent/albelian/one dimensional the quadratures take successively simpler forms. We discuss this point in more detail later on.

Let $\mathscr{H}$ be the Lie algebra of vector fields generated by $\{h(z,v):v=\text{constant}\}$. In [10] it is shown that a necessary condition for (1.4) to simulate (1.2) is that $\mathscr{F}$ is the homomorphic image of a subalgebra of $\mathscr{H}$. Moreover if $\mathscr{H}$ is finite dimensional then this condition is almost sufficient in the following sense. If $\mathscr{F}$ is the homomorphic image of a subalgebra of $\mathscr{H}$ then (1.4) may not simulate (1.2) but one can construct a system with algebra isomorphic to $\mathscr{H}$ which does simulate (1.2).

But we wish to focus on the complexity of (1.1) as introduced by the stochastics. The solution satisfies the stochastic integral equation

$$x(t)=x^0+\int_0^t f(x(s))\,ds+\int_0^t g(x(s))\,dW(s). \tag{1.5}$$

Can this be reformulated as a series of stochastic quadratures and a nonstochastic integral equation? Put another way can (1.1) be simulated by the composition of two systems, one of which involves the stochastics but is solvable by quadratures and the other which is more complicated but has no white noise driving term, dW? An answer to these questions is supplied by the following. We denote by $\mathscr{G}$ the Lie algebra of vector fields generated by $\{g^1,\ldots,g^m\}$ and for convenience we assume that the initial condition X^0 of (1.1b) is equal to x^0 almost surely.

THEOREM 1. *Suppose G is of finite dimension d. Then there exists a neighborhood $\mathscr{U}$ of the $(y^0,z^0)\in\mathbb{R}^d\times\mathbb{R}^n$*

i) *a stochastic differential equation*

$$dY=k(Y)\,dW \tag{1.6a}$$

$$Y(0)=y^0 \tag{1.6b}$$

$$dZ=h(Y,Z)\,dt \tag{1.6c}$$

$$Z(0)=z^0 \tag{1.6d}$$

where Y and Z are of dimensions d and n and $k(y)$, $h(y,z)$ are smooth functions of dimensions $d\times m$ and $n\times 1$ for $(y,z)\in\mathscr{U}$,

ii) *a smooth map $\varphi(y,z)=x$ defined for $(y,z)\in\mathscr{U}$ and*

iii) *a stopping time $T>0$ a.s. such that if $(Y(t),Z(t))$ and $X(t)$ are the Stratonovich solutions of (1.6) and (1.1) then for $0\leq t<T$*

$$\varphi(Y(t),Z(t))=X(t) \tag{1.7}$$

Moreover φ induces an isomorphism from Lie algebra $\mathscr{K}$ generated by the columns of k onto $\mathscr{G}$.

After changing the vector field h, the result also holds for the Ito solutions of (1.6) and (1.1).

We defer the proof of this and other results to the next section. Notice that the stochastic complexity of solving (1.6) is wholly contained in the first part (1.6a, b). For a solution of this will have continuous sample paths almost surely and hence (1.6c, d) can be solved for each sample path individually using standard techniques. Hence (1.1) can be solved for each sample path individually if (1.6a, b) can.

Actually we have considerable freedom in our choice of the stochastic differential equation (1.6a, b). This is advantageous for as we shall see in a moment if the Lie algebra $\mathscr{G}$ is not too complicated then by an appropriate choice of (1.6a, b) we can greatly simplify the stochastic integrations required to solve the equation.

THEOREM *Let $\mathscr{G}$ be of finite dimension d and $k^1(y), \ldots, k^m(y)$ be smooth vector fields defined on a neighborhood of $y^0 \in \mathbb{R}^l$. Let $\mathscr{K}$ denote the Lie algebra generated by these vector fields, $\mathscr{K}(y)$ the linear space spanned by the vector fields of $\mathscr{K}$ evaluated at y and $\mathscr{K}^y$ the isotropy subalgebra at y, i.e., the space of vector fields of $\mathscr{K}$ which vanish at y. Assume $\mathscr{K}$ acts transitively, i.e., $\mathscr{K}(y) = \mathbb{R}^l$ and freely, i.e., $\mathscr{K}^y = \{0\}$ for every y in some neighborhood of y^0. Finally assume the correspondence $k^j(y) \mapsto g^j(x)$ extends to a Lie algebra homomorphism of $\mathscr{K}$ onto. $\mathscr{G}$. Then Theorem 1 holds with $k^1(y), \ldots, k^m(y)$ defining (1.6a, b).*

Let us see how these theorems lead to some of the recent results regarding stochastic differential equations which have appeared in the literature. First we consider the case of a scalar diving noise, $m = 1$. This has been treated by Lamperti [12], Sussmann [13], Doss [5] and Krener [11]. In this case the Lie algebra $\mathscr{G}$ is one dimensional, and hence isomorphic to any one dimensional algebra. We apply Theorem 2 with y a scalar, $k^1(y) = 1$, the constant vector field and $y^0 = 0$. Then (1.6a, b) becomes the trivial stochastic differential equation

$$dY = dW$$
$$Y(0) = 0$$

whose solution is $Y(t) = W(t)$. We next solve (1.6c, d) for each sample path independently. This is possible because the sample paths of Wiener process are almost surely continuous. If $w(t)$ is a particular sample path then we must solve the ordinary differential equation

$$\frac{dz}{dt}=h(z,w(t))$$
$$z(0)=z^0$$

which is certainly possible since the right hand side is smooth in z and continuous in t. Finally we plug this into the mapping φ to get $x(t)=\varphi(z(t),w(t))$, the sample path solution of (1.1).

Suppose there are several noises and the Lie algebra $\mathcal{G}$ is abelian. This case has been discussed by Doss [5], Clark [2] and Davis [3,4]. We proceed as before, applying Theorem 2 with the dimension of y equal to m, $k^j(y)$ the constant unit vector field pointing in the jth direction and $y^0=0$. Again (1.6a, b) is trivial

$$dY_1=dW_i \quad i=1,\ldots,m$$
$$Y_i(0)=0$$

and as before we can construct the solution sample path by sample path.

Recently, Yamato [15] discussed the case where the Lie algebra $\mathcal{G}$ is nilpotent. By Ado's theorem every finite dimensional Lie algebra has a faithful representation as a matrix algebra. Therefore by applying this to Theorem 2, we can choose the vector fields $k^1(y),\ldots,k^m(y)$ to be linear in y

$$k^j(y)=A^j y$$

for some $l\times l$ matrices $A^1,\ldots,A^m$ which generated a Lie algebra isomorphic to $\mathcal{G}$. If $\mathcal{G}$ is nilpotent then, in the appropriate coordinates, the matrices $A^1,\ldots,A^m$ are strictly lower triangular. If $A^j=(a^j_{ik})$ then (1.6a, b) becomes

$$dY_i=\sum_{j=1}^{m}\sum_{k=1}^{i-1} a^j_{ik}\, Y_k\, dW_j \tag{1.8a}$$
$$Y_i(0)=y_i^0 \tag{1.8b}$$

for $i=1,\ldots,l$. These equations are solvable in terms of multiple quadratures of the form

$$\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-1}} dW_{j_n}(s_n)\,dW_{j_{n-1}}(s_{n-1})\ldots dW_{j_n}(s_1). \tag{1.9}$$

For deterministic systems, expressions such as (1.9) are called *multiple path integrals* by K. T. Chen. They have been used extensively by Fliess [6,7] in treating deterministic systems. If the integrations of (1.9) are to be interpreted in the Ito sense then Ito calls them *multiple Wiener integrals* [8]. With Stratonovich interpretation they are closer to the multiple integrals found in Wiener's famous paper, *The Homogeneous Chaos* [14].

If $\mathscr{G}$ is solvable then the situation is similar except that the matrices $A^1, \ldots, A^m$ need only be lower triangular. Therefore (1.6a, b) becomes

$$dY_i = \sum_{j=1}^{m} \sum_{k=1}^{i} a_{ik}^j \, Y_k \, dW_j \tag{1.10a}$$

$$Y_i(0) = y_i^0 \tag{1.10b}$$

for $i = 1, \ldots, l$. By the variation of parameters formula we obtain the Stratonovich solution

$$Y_i(t) = \exp\left(\sum_{j=1}^{m} a_{ii}^j W_j(t)\right) \times \left[y_i^0 + \int_0^t \exp\left(- \sum_{j=1}^{m} a_{ii}^j \, W_j(s)\right) \sum_{k=1}^{i-1} \sum_{j=1}^{m} a_{ik}^j \, Y_k(s) \, dW_j(s) \right]. \tag{1.11}$$

The Ito solution is a bit more complicated

$$Y_i(t) = \exp\left(\sum_{j=1}^{m} a_{ii}^j (W_j(t) - a_{ii}^j t/2)\right) \times \left[y_i^0 + \int_0^t \exp\left(- \sum_{j=1}^{m} a_{ii}^j \, (W_j(s) - a_{ii}^j s/2)\right) \sum_{k=1}^{i-1} \sum_{j=1}^{m} + a_{ik}^j \, Y_k(s) (dW_j(s) - a_{ii}^j \, ds) \right]. \tag{1.12}$$

Both solutions (1.11) and (1.12) can be expressed in terms of multiple stochastic integrals of the form

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} d\exp(\alpha_{j_n} W_{j_n}(s_n)) \ldots d\exp(\alpha_{j_1} W_{j_1}(s_1)). \tag{1.13}$$

This is similar to the multiple Wiener integrals (1.9). The expression of the Ito solution (1.12) also involves multiple integrals of mixed stochastic and deterministic type.

In closing this section we would like to emphasize that Theorems 1 and 2 do not imply that the deterministic complexity of the differential equation has been lessened by replacing (1.1) by (1.6). It is only that the intrinsic stochastic complexity has been separated out into (1.6a, b). For a full treatment of decompositions of differential systems we refer the reader to [10].

Since completing the first version of this paper, similar results of Kunita's have come to our attention [16].

2. PROOFS

In this section we prove Theorems 1 and 2 for both the Stratonovich and Ito interpretations of (1.1). We start with the Stratonovich version of Theorem 1 and then modify it to obtain the others. The proofs involve generalizations of techniques used by Sussmann, Doss and Yamato.

Let $\tilde{g}^1(x), \ldots, \tilde{g}^d(x)$ be a basis for $\mathscr{G}$; then

$$g^j(x) = \sum_{j=1}^{d} b_i^j \tilde{g}^i(x) \tag{2.1}$$

for $j = 1, \ldots, m$. Let $\{c_k^{ij}\}$ be the structural constants of $\mathscr{G}$ relative to this basis

$$[\tilde{g}^i, \tilde{g}^j](x) = \sum_{k=1}^{d} c_k^{ij} \tilde{g}^k(x) \tag{2.2}$$

Let $\gamma^j(t)x$ denote the flow of $\tilde{g}^j(x)$, i.e.,

$$\frac{d}{dt}\gamma^j(t)x = \tilde{g}^j(\gamma^j(t)x) \tag{2.3a}$$

$$\gamma^j(0)x = x. \tag{2.3b}$$

The juxtaposition of $\gamma^j(t)$ with x does not mean multiplication; we use this notation instead of $\gamma^j(t;x)$ to avoid an excess of parentheses. Let $\gamma^j(t)_*$ denote the tangent map,

$$\gamma^j(t)_* \tilde{g}^i(x) = \frac{\partial \gamma^j(t)}{\partial x}(x)\tilde{g}^i(x).$$

We make use of the following lemma.

LEMMA. *Let $\lambda_k^{ij}(t)$ be the analytic functions defined as the solutions of the differential equations*

$$\frac{d}{dt}\lambda_k^{ij} = \sum_{n=1}^{d} c_n^{ij} \lambda_k^{nj} \tag{2.4a}$$

$$\lambda_k^{ij}(0) = \delta_k^i \tag{2.4b}$$

for $i, j, k=1,\ldots,d$ *where* δ_k^i *is the Kroenecker* δ. *Then for any* x *and* $i, j=1,\ldots,d$

$$\gamma^j(t)_* \tilde{g}^i(\gamma^j(-t)x) = \sum_{k=1}^{d} \lambda_k^{ij}(t)\tilde{g}^k(x). \tag{2.5}$$

Proof of Lemma. From the group property of flows,

$$\gamma^j(t+s)x = \gamma^j(t)\gamma^j(s)x = \gamma^j(s)\gamma^j(t)x,$$

it follows that

$$\frac{d}{dt}\gamma^j(t)_* \tilde{g}^i(\gamma^j(-t)x) = \gamma^j(t)_* \frac{d}{ds}\bigg|_{s=0} \gamma^j(s)_* \tilde{g}^i(\gamma^j(-s)\gamma^j(-t)x)$$

It is a straightforward calculation [1, p. 17] that

$$\frac{d}{ds}\bigg|_{s=0} \gamma^j(s)_* \tilde{g}^i(\gamma^j(-s)x) = [\tilde{g}^i, \tilde{g}^j](x) = \sum_{n=1}^{d} c_n^{ij}\tilde{g}^n(x)$$

so

$$\frac{d}{dt}\gamma^j(t)_* \tilde{g}^i(\gamma^j(-t)x) = \sum_{n=1}^{d} c_n^{ij}\gamma^j(t)_* g^n(\gamma^j(-t)x). \tag{2.6}$$

Differentiating the other side of (2.5) we obtain

$$\frac{d}{dt}\left(\sum_{k=1}^{d} \lambda_k^{ij}(t)\tilde{g}^k(x)\right) = \sum_{k=1}^{d}\sum_{n=1}^{d} c_n^{ij}\lambda_k^{nj}\tilde{g}^k(x). \tag{2.7}$$

Notice that at any t where (2.5) holds the derivatives (2.6) and (2.7) of both sides agree. Clearly (2.5) holds at $t=0$ so by uniqueness of solutions to differential equations the lemma is proved.

For simplicity of notation assume x^0, y^0 and z^0 are the origins in $\mathbb{R}^n$, $\mathbb{R}^d$ and $\mathbb{R}^n$. For small $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^n$ define a map $\varphi(y,z)=x$ by

$$\varphi(y,z) = \gamma^1(y_1)\ldots\gamma^d(y_d)z. \tag{2.8}$$

Then

$$\frac{\partial\varphi}{\partial y_j}(y,z) = \gamma^1(y_1)_* \cdots \gamma^{j-1}(y_{j-1})_* \tilde{g}^j(\gamma^j(y_j)\ldots\gamma^1(y_d)x)$$

$$= \gamma^1(y_1)_* \cdots \gamma^{j-1}(y_{j-1})_* \tilde{g}^j(\gamma^{j-1}(-y_{i-1})\ldots\gamma^1(-y_1)\varphi(y_1 z))$$

By repeated use of the lemma we conclude that there exists analytic functions $\beta_i^j(y)$ such that

$$\frac{\partial \varphi}{\partial y_j}(y,z)=\sum_{i=1}^{d}\beta_i^j(y)\tilde{g}^i(\varphi(y,z)) \tag{2.9a}$$

$$\beta_i^j(0)=\delta_i^j \tag{2.9b}$$

From this we see that the matrix $(\beta_i^j(y))$ is invertible for small y. Let $\tilde{k}(y)$ denote the inverse matrix and $\tilde{k}^j(y)$, the jth column of $\tilde{k}$, we view $\tilde{k}^j(y)$ as a vector field. By construction $\tilde{k}^j(y)$ is φ-related to $\tilde{g}^j(x)$, i.e., the Jacobian of φ maps $\tilde{k}^j(y)$ to $\tilde{g}^j(x)$

$$\frac{\partial \varphi}{\partial y}(y,z)\tilde{k}^j(y)=\tilde{g}^j(\varphi(y,z)) \tag{2.10}$$

for all small y and z. This implies [1, p. 14] the brackets of the $\tilde{k}^j$'s are φ-related to the corresponding brackets of $\tilde{g}^j$'s. In other words the Jacobian of φ is a homomorphosm from the Lie algebra $\mathscr{K}$ generated by $\tilde{k}^1,\ldots,\tilde{k}^d$ onto $\mathscr{G}$.

Actually this mapping is an isomorphism for if there exists a vector field $k(y)\in\mathscr{K}$ such that for some small y and all small z

$$\frac{\partial \varphi}{\partial y}(y,z)k(y)=0$$

then for all small $x=\varphi(y,z)$

$$\frac{\partial \varphi}{\partial y}(y,z)k(y)=\sum_{i,j=1}^{d}\beta_i^j(y)k_j(y)\tilde{g}^i(x)=0.$$

But this contradicts the assumption that $\tilde{g}^1(x),\ldots,\tilde{g}^m(x)$ is a basis for $\mathscr{G}$.

Next we define

$$k^j(y)=\sum_{i=1}^{d}b_i^j\tilde{k}^i(y)$$

then by (2.1) and (2.10) for small y and z.

$$\frac{\partial \varphi}{\partial y}(z,y)k^j(y)=g^j(\varphi(y,z)) \tag{2.11}$$

By the uniqueness of solutions to ordinary differential equations we conclude that the $n\times m$ matrix $\partial\varphi/\partial z(y,z)$ is invertible. Let $\alpha(y,z)$ denote the inverse matrix and define the vector field $h(y,z)$ by

$$h(y,z)=\alpha(y,z)f(\varphi(y,z)).$$

The h is φ-related to f,

$$\frac{\partial\varphi}{\partial z}(y,z)h(y,z)=f(\varphi(y,z)). \tag{2.12}$$

We use this $h(y,z)$ and the above $k^1(y),\ldots,k^m(y)$ to construct (1.6) on some open neighborhood $\mathscr{U}$ of $(0,0)$ where φ is also defined. Let $(Y(t),Z(t))$ be the Stratonovich solution of (1.1) and T the stopping time

$$T=\inf\{t>0:(Y(t),Z(t))\notin\mathscr{U}\}.$$

Using the standard differential rule, (2.11) and (2.12) it is straightforward to verify that $X(t)$ defined as $\varphi(Y(t),Z(t))$ is Stratonovich solution of (1.1) for $0<t<T$.

When (1.1) and (1.6) are to be interpreted in the Ito sense, we define the vector field h by

$$h(y,z)=\alpha(y,z)\left[f(\varphi(y,z))-1/2\sum_{i,j=1}^{d}\sum_{n=1}^{m}\frac{\partial^2\varphi}{\partial y_i\partial y_j}k_i^n(y)k_j^n(y)\right] \tag{2.13}$$

so that

$$f(\varphi(y,z))=\frac{\partial\varphi}{\partial z}(y,z)h(y,z)+1/2\sum_{i,j=1}^{d}\sum_{n=1}^{m}\frac{\partial^2\varphi}{\partial y_i\partial y_j}(y,z)k_i^n(y)k_j^n(y). \tag{2.14}$$

Using the Ito differential rule, (2.11) and (2.14) it is straightforforward to verify that if $(Y(t),\ Z(t))$ is the Ito solution of (1.6) then $X(t)=\varphi(Y(t),Z(t))$ is the Ito solution of (1.1).

Moving on to the proof of Theorem 2, we choose vector fields $\tilde{k}^1(y),\ldots\tilde{k}^l(y)$ which span $\mathscr{K}$. Since the action is free these vector fields evaluated at any small y span $\mathbb{R}^l$. Let $\tilde{g}^1(x),\ldots,\tilde{g}^l(x)$ be the homomorphic images of these vector fields, and let $\varkappa^1(t)y,\ldots,\varkappa^l(t)y$; $\gamma^1(t)x,\ldots,\gamma^l(t)x$ be their flows. Let $s=(s_1,\ldots,s_l)\in\mathbb{R}^l$ and $z\in\mathbb{R}^n$. Consider the maps

$$s\mapsto y(s)=\varkappa^1(s_1)\ldots\varkappa^l(s_l)y^0$$

$$(s,z)\mapsto x(s,z)=\gamma^1(s_1)\ldots\gamma^l(s_l)z$$

defined for small s and z. The first is invertible, let $s(y)$ denote its inverse.

We define

$$\varphi(y, z) = x(s(y), z)$$

then φ_* is a Lie algebra homomorphism from $\mathscr{K}$ onto $\mathscr{G}$ and $\tilde{k}^j(y)$ is φ-related to $\tilde{g}^j(x)$. (See Krener [9], [10] for details.) The rest of the proof proceeds essentially as with Theorem 1.

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