Locally (f, g) invariant distributions *

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Conditions are derived for the existence of (f, g) invariant distributions in nonlinear systems and a procedure for the construction of the corresponding feedback control law is given.

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1. Introduction

In the study of decoupling of linear systems the important geometric concept is that of (A, B) invariance [1]. Recall that for the linear system

$$\dot{x} = Ax + Bu \tag{1.1}$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$, a subspace $\mathcal{V} \subseteq \mathbf{R}^n$ is (A, B) invariant if there exists an $m \times n$ matrix F defining a linear feedback law $u = Fx + v$ such that the modified dynamics

$$\dot{x} = (A + BF)x + Bv \tag{1.2}$$

leaves $\mathcal{V}$ invariant, i.e.

$$(A + BF)\mathcal{V} \subseteq \mathcal{V}. \tag{1.3}$$

It is well known and easy to see that (1.3) is equivalent to

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{R}(B) \tag{1.4}$$

where $\mathcal{R}(B)$ denotes the subspace spanned by the columns of B.

A similar concept arises in the decoupling of nonlinear systems as we discussed in [2].

Consider the nonlinear system

$$\dot{x} = f(x,u) = g^0(x) + g(x)u$$
$$= g^0(x) + \sum_{j=1}^{m} g^j(x)u \tag{1.5}$$

where x are local coordinates of a smooth n-dimensional manifold M, $u \in \mathbf{R}^m$, g^0 and the m columns g^j of g are smooth vector fields on M (smooth means either C^∞ or analytic). A distribution Δ on M is (f, g) *invariant* if there exists a nonlinear feedback law $u = \alpha(x) + \beta(x)v$, when α and β are smooth functions taking values in $\mathbf{R}^m$ and $\mathbf{R}^{m\times m}$, such that the modified dynamics

$$\dot{x} = g^0(x) + g(x)\alpha(x) + g(x)\beta(x)v \tag{1.6}$$

leaves Δ invariant, i.e.

$$[g^0 + g\alpha, \Delta](x) \subseteq \Delta(x), \tag{1.7a}$$

$$[g\beta, \Delta](x) \subseteq \Delta(x). \tag{1.7b}$$

In general, it is desirable to maintain as much open loop control as possible; therefore, one seeks a $\beta(x)$ satisfying (1.7b) which is invertible. (Condition (1.7b) is trivially satisfied in the linear case since the open loop term Bv does not depend on x.)

It is easy to see that (1.7) implies that

$$[g^0, \Delta](x) \subseteq \Delta(x) + \mathcal{R}(g(x)), \tag{1.8a}$$

$$[g, \Delta](x) \subseteq \Delta(x) + \mathcal{R}(g(x)); \tag{1.8b}$$

however, the reverse implication does not always hold. ($\mathcal{R}(g(x))$ is the span of the column of $g(x)$.) A stronger form of (1.8b) is

$$[g, \Delta](x) \subseteq \Delta(x) \tag{1.8c}$$

which yields the following partial converse:

Hirschorn's Lemma [3]. *Suppose Δ is involutive and the dimensions of $\Delta(x)$, $\mathcal{R}(g(x))$ and $\Delta(x) \cap \mathcal{R}(g(x))$ are constant over* M. *If* (1.8a) *and* (1.8c) *hold, then locally around each x there exists an $\alpha(x)$*

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satisfying (1.7a). *If M is simply connected then a global $\alpha(x)$ can be found.*

A similar result can be found in [4].

In [2] we defined a locally (f,g) invariant distribution as one which satisfies (1.8a) and (1.8b) and stated without proof a weaker version [2; Lemma 2.3] of the following lemma:

Lemma. *Suppose Δ is a locally (f,g) invariant distribution, $\bar{\Delta}$ its involutive closure, and the dimensions of $\bar{\Delta}(x)$, $\mathcal{R}(g(x))$ and $\bar{\Delta}(x)\cap\mathcal{R}(g(x))$ are constant. Then locally around each x there exists $\alpha(x)$ and invertible $\beta(x)$ satisfying* (1.7a) *and* (1.7b).

The purpose of this letter is to give the proof of this.

After submitting this note for publication, it has come to our attention that similar results are found in an unpublished manuscript of Nijmeijer [5].

2. Proof of the lemma

We prove the existence of α and β by showing the integrability of certain first order partial differential equations given by (1.8). A smooth one form ω on M defines a first order partial differential equation. Given ω, we seek a function $\varphi: M \to \mathbf{R}$ such that

$$d\varphi = \omega, \tag{2.1a}$$

$$\varphi(x^0) = \varphi^0. \tag{2.1b}$$

The fundamental result is the following:

Proposition 1. *A solution φ of* (2.1) *exists locally around x^0 iff in a neighborhood of x^0 ω is a closed one form, i.e. $d\omega = 0$. If the solution exists, it is unique. If M is simply connected and ω is closed, then a unique global solution exists.*

One can describe a partial differential equation in a dual fashion. Suppose $X^1,\ldots,X^n$ are smooth vector fields which are linearly independent at each point and $\gamma^1,\ldots,\gamma^n$ are smooth functions. We seek a solution φ of

$$X^k(\varphi) = \gamma^k, \tag{2.2a}$$

$$\varphi(x^0) = \varphi^0. \tag{2.2b}$$

(The vector fields and functions of (2.2a) need not be globally defined. Rather for each element of an open cover of M, an equation of the form (2.2a) might be given subject to the obvious compatibility conditions on the intersections of the open sets.) The following is the equivalent formulation of Proposition 1:

Proposition 2. *A solution φ of* (2.2) *exists locally around x^0 iff in a neighborhood of x^0*

$$X^i(\gamma^j) - X^j(\gamma^i) = \sum_{k=1}^{n} C_k^{ij}\gamma^k \tag{2.3}$$

where C_k^{ij} are structural coefficients defined by

$$[X^i, X^j] = \sum_{k=1}^{n} C_k^{ij} X^k.$$

If the solution exists, it is unique. If (2.3) *is globally satisfied and M is simply connected, then a unique global solution exists.*

The equivalence of these follows immediately from the identity

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Suppose not all the partial derivatives of φ are specified; instead, for some $d \leqslant n$ we seek a solution of (2.2a) for $k = 1,\ldots,d$. It is convenient to assume the distribution Δ spanned by $\{X^1,\ldots,X^d\}$ is involutive for if $[X^i, X^j]$ is not in Δ, let $X^{d+1} = [X^i, X^j]$ and $\gamma^{d+1} = X^i(\gamma^j) - X^j(\gamma^i)$. Then φ must also satisfy the additional constraint

$$X^{d+1}(\varphi) = \gamma^{d+1}$$

and this can be incorporated into (2.2a) by increasing d by 1.

To specify a unique solution of (2.2a) for $i = 1,\ldots,d \leqslant n$, we must specify the value of φ on an $n-d$ dimensional submanifold which is complementary to Δ. A submanifold N is *complementary* to Δ if for each $x \in N$ T_xN and $\Delta(x)$ are complementary subspaces of T_x, i.e.

$$T_xN + \Delta(x) = T_xN,$$

$$T_xN \cap \Delta(x) = \{0\}.$$

Given such a manifold N we repace (2.2b) by the boundary condition

$$\varphi(x) = \psi(x) \quad \forall x \in N \tag{2.2c}$$

where $\psi(x)$ is a smooth function defined on N. Of course, if N is complementary to Δ, this implies that the rank of $\Delta(x)$ is constant around N, and since Δ is involutive, by Frobenius' theorem it is integrable.

Proposition 3. *Consider the partial differential equation (2.2a) subject to the boundary condition (2.2c), where $\{X^1,\dots,X^d\}$ span an involutive distribution Δ complementary to N and $\gamma^1,\dots,\gamma^d$ are smooth functions defined around N, ψ is a smooth function on N. There exists a unique solution locally around $x^0 \in N$ iff the integrability conditions*

$$X^i(\gamma^j) - X^j(\gamma^i) = \sum_{i=1}^{d} C_k^{ij} X^k \tag{2.4}$$

are satisfied around $x^0 \in N$. Again C_k^{ij} are the structural coefficients of Δ,

$$[X^i, X^j] = \sum_{k=1}^{d} C_k^{ij} X^k.$$

If Δ is a regular distribution so that $\pi: M \to M/\Delta$ is a fiber bundle, N is a section of this bundle, the integral manifolds of Δ are simply connected, and the integrability conditions are satisfied, then a unique global solution exists.

This proposition is proved by solving the partial differential equation on the leaves of Δ, obtaining solutions from Proposition 2. That these solutions fit together nicely to obtain a smooth solution on M follows from the continuous dependence of solutions of ordinary differential equations on initial conditions.

We now turn to the proof of the lemma. Assume Δ is locally (f,g) invariant; we first show that so is $\bar{\Delta}$, its involutive closure. Define a sequence of distributions Δ^k by $\Delta^0 = \Delta$ and

$$\Delta^{k+1} = \Delta^k + [\Delta, \Delta^k].$$

Then $\bar{\Delta} = \bigcup_{k \geq 0} \Delta^k$. We show by induction that each Δ^k is locally (f,g) invariant, hence implying $\bar{\Delta}$ is also. By assumption $\Delta^0 = \Delta$ is locally (f,g) invariant, assume Δ^k is. Let $X^1 \in \Delta$ and $X^2 \in \Delta^k$, then there exist $m \times m$-matrices Γ^1, Γ^2 and $Y^1 \in \Delta$, $Y^2 \in \Delta^k$ (each column of Y^i a vector field of Δ or Δ^k) such that

$$[g, X^i] = g\Gamma^i + Y^i.$$

Using the Jacobi identity

$$\begin{aligned}
[g[X^1, X^2]] &= [[g, X^1] X^2] - [[g, X^2] X^2] \\
&= [g\Gamma^1 + Y^1, X^2] - [g\Gamma^2 + Y^2, X^1] \\
&= g\Gamma^2\Gamma^1 + Y^2\Gamma^1 - gX^2(\Gamma^1) + [Y^1, X^2] \\
&\quad - g\Gamma^1\Gamma^2 - Y^1\Gamma^2 + gX^1(\Gamma^2) - [Y^2, X^1] \\
&\in \Delta^k + [\Delta, \Delta^k] + \mathcal{R}(g).
\end{aligned}$$

A similar calculation holds for g^0.

In light of the above we can assume $\Delta = \bar{\Delta}$ is involutive and locally has a basis $\{X^1,\dots,X^d\}$. After modification of g by feedback, we can assume that $g^1(x),\dots,g^l(x)$ are linearly independent and

$$\Delta(x) \cap \mathcal{R}(g(x)) = \mathcal{R}\big(g^{l+1}(x),\dots,g^m(x)\big), \tag{2.5a}$$

$$\Delta(x) \cap \mathcal{R}\big(g^1(x),\dots,g^l(x)\big) = \{0\}. \tag{2.5b}$$

From (1.8b) we have the existence of γ_i^{jk} such that

$$[g^j, X^k] \equiv \sum_{i=1}^{m} \gamma_i^{jk} g^i \mod \Delta \tag{2.6a}$$

for $i,j = 1,\dots,m$, $k = 1,\dots,d$. To uniquely fix the γ_i^{jk} we specify $\gamma_i^{jk} = 0$ if i or j equals $l+1,\dots,m$. Let Γ^k be the $m \times m$ matrix with i-j entry γ_i^{jk}, then

$$[g, X^k] \equiv g\Gamma^k \mod \Delta. \tag{2.6b}$$

The $m \times m$ feedback matrix β will satisfy (1.7) if

$$[g\beta, X^k] \equiv 0 \mod \Delta \tag{2.7}$$

or

$$g\big(\Gamma^k\beta - X^k(\beta)\big) \equiv 0 \mod \Delta.$$

Therefore we seek a solution of

$$X^k(\beta) = \Gamma^k\beta \quad \text{for } k = 1,\dots,d. \tag{2.8}$$

From Proposition 3 we see that a solution will exist locally if the integrability conditions are satisfied and we can find a complementary submanifold on which to set boundary conditions. The latter poses no problem for by Frobenius' theorem around x^0 we can choose local coordinates x such that $x^0 = 0$ and the integral manifolds of Δ are given by $x_{d+1} = \text{constant},\dots,x_n = \text{constant}$. For N we take the submanifold given locally by $x_1 = 0,\dots,x_d = 0$ and impose the boundary conditions

$\beta(x) = I^{m\times m} \quad \forall x \in N.$

Next we check the integrability condition

$$X^i(\Gamma^j\beta) - X^j(\Gamma^i\beta) = \sum_{k=1}^{d} C_k^{ij}\Gamma^k\beta$$

which is implied by

$$\Gamma^j\Gamma^i + X^i(\Gamma^j) - \Gamma^i\Gamma^j - X^j(\Gamma^i) = \\ = \sum_{k=1}^{d} C_k^{ij}\Gamma^k. \qquad (2.9)$$

The Jacobi identity yields

$$[[g, X^i]X^j] - [[g, X^j]X^i] = [g[X^i, X^j]]$$

which we compute mod Δ using (2.6b) and the involutiveness of Δ:

$$[g\Gamma^i, X^j] - [g\Gamma^j, X^i] \equiv \left[g, \sum_{k=1}^{d} C_k^{ij}X^k\right]$$

or

$$g\left(\Gamma^j\Gamma^i - X^j(\Gamma^i) - \Gamma^i\Gamma^j + X^i(\Gamma^j)\right) \equiv \\ \equiv g\left(\sum_{k=1}^{d} C_k^{ij}\Gamma^k\right).$$

The first l rows of the matrices multiplying g on each side must agree by (2.5), the last $m-l$ rows are zero since $\gamma_i^{jk} = 0$ for $i = l+1,\ldots,m$, hence (2.9) holds.

Notice that β is invertible wherever it is defined, for (2.8) is just a linear ordinary differential equation along an integral curve of X^k, therefore β does not change rank along such curves. The totality of these curves emanating from N fill up an open neighborhood of x^0; therefore the rank is everywhere equal to m.

To show the existence of α we merely need to add a zero row and column to β of the above, with $\alpha_i = \beta_i^0$, $i = 1,\ldots,m$. If we initialize $\beta_i^j = \delta_i^j$ as before, then since $\gamma_0^{jk} = 0$ as defined by (2.6a) we see that $\beta_0^j(x) = \delta_0^j$ for all x as desired.

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